# Mathematical Reviews

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# Mathematical Reviews

Vol. 10, No. 6 JUNE, 1949 Pages 347-418

## ALGEBRA

**Rohrbach, Hans. Eine Bemerkung zu einer Arbeit von H. Hadwiger.** Mitt. Verein. Schweiz. Versich.-Math. 48, 43–45 (1948).

Proof of equivalence of results on the number of permutations of $n$ elements without any of the successions 12, 23, $\cdots$, $(n-1)n$, reached previously by H. Hadwiger [Mitt. Verein. Schweiz. Versich.-Math. 46, 105–109 (1946); these Rev. 7, 406] and by Armsen and the author [Ber. Math.-Tagung Tübingen 1946, pp. 36–37 (1947); these Rev. 9, 2].

*J. Riordan* (New York, N. Y.).

**Armsen, Paul, und Rohrbach, Hans. Sequenzen in Permutationen.** Arch. Math. 1, 106–112 (1948).

A more detailed treatment of the results reported in the paper and references of the preceding review.

*J. Riordan* (New York, N. Y.).

**Pizá, Pedro A. Kummer coefficients.** Revista Unión Mat. Argentina 13, 125–130 (1948). (Spanish)

The Kummer coefficients $K(n, m)$ in question are those of the expansion $x^n+y^n=\sum(-1)^m K(n, m)(xy)^m(x+y)^{n-2m}$. The author infers certain properties from a table of values extending to $n=23$ without recognition of the known result [cf. Chrystal, Algebra, part II, London, 1900, p. 201]:

$$K(n, m)=n(n-m)^{-1}\binom{n-m}{m}=\binom{n-m}{m}+\binom{n-m-1}{m-1}.$$

*J. Riordan* (New York, N. Y.).

**Babini, José. Note on the Kummer coefficients.** Revista Unión Mat. Argentina 13, 131–134 (1948). (Spanish)

The conjectures of the preceding paper are verified by noting the relation of the numbers to binomial coefficients and finding certain generating functions. Among the last the following may be noted:

$$\sum_0^\infty K(n+m, m)x^m=(1+2x)(1+x)^{n-1},$$

$$\sum_0^\infty K(n+2m, m)x^m=(1+x)(1-x)^{-n-1}.$$

*J. Riordan* (New York, N. Y.).

**Kerawala, S. M. A note on self-conjugate Latin squares of prime degree.** Math. Student 15 (1947), 16 (1948).

The author provides a counter example to a conjecture of S. M. Jacob [Proc. London Math. Soc. (2) 31, 329–354 (1930)] about self-conjugate Latin squares of prime degree.

*H. B. Mann* (Columbus, Ohio).

**Rollero, Aldo. Sulla potenza ad esponente intero positivo di un determinante.** Atti Accad. Ligure 4 (1947), 26–31 (1948).

The author expresses the $k$th power of a determinant of order $n$, $k$ a positive integer, as a determinant of order $n$.

*C. C. MacDuffee* (Madison, Wis.).

**Taussky, O., and Wigglesworth, L. A. Note on a theorem in $n$-dimensional geometry.** Amer. Math. Monthly 55, 492–494 (1948).

This is a proof, partly geometric, of a theorem of Schläfli [Denkschr. Akad. Wiss. Wien. Abt. 2. 4, 1–74 (1852)]: If $\Delta_s$ is the determinant of the system of linear homogeneous equations

$$(x_{i1}U_1+\cdots+x_{in}U_n)(x_{j1}U_1+\cdots+x_{jn}U_n)=0 \qquad (i, j=1, \cdots, n)$$

in the unknowns $U_kU_l$, then $\Delta_s\neq 0$ if $|x_{ik}|\neq 0$.

*C. C. MacDuffee* (Madison, Wis.).

**Gårding, Lars. Extension of a formula by Cayley to symmetric determinants.** Proc. Edinburgh Math. Soc. (2) 8, 73–75 (1948).

Let $x=\det(x_{ik})$ be a symmetric $n$-rowed determinant whose elements are independent variables. Let $\xi=\det(\xi^{ik})$ have as elements the operators $\xi^{ii}=\partial/\partial x_{ii}$, $\xi^{ik}=\frac{1}{2}\partial/\partial x_{ik}$, $i\neq k$. Let $x_{i_1\ldots i_m k_1\ldots k_m}$ be an $m$-rowed minor of $x$, $1\leqq m\leqq n$, and let $x^{i_1\cdots i_m k_1\cdots k_m}$ be its algebraic complement. It is shown that

$$\xi^{i_1\cdots i_m k_1\cdots k_m}x^\alpha=\alpha(\alpha+\tfrac{1}{2})\cdots(\alpha+\tfrac{1}{2}(m-1))x^{\alpha-1}x^{i_1\cdots i_m k_1\cdots k_m}.$$

*C. C. MacDuffee* (Madison, Wis.).

**Turnbull, H. W. Symmetric determinants and the Cayley and Capelli operators.** Proc. Edinburgh Math. Soc. (2) 8, 76–86 (1948).

Enlarging upon the above paper by Gårding, the author similarly modifies the Capelli operator [A. Capelli, Math. Ann. 29, 331–338 (1887)] so that it may be applied to symmetric determinants. There is also a discussion of the relation between the identities of Sylvester and those of Kronecker and of their generalizations.

*C. C. MacDuffee* (Madison, Wis.).

**Turnbull, H. W. Note upon the generalized Cayleyan operator.** Canadian J. Math. 1, 48–56 (1949).

Let $x_{ij}$ be independent variables. Let $\Delta=(x_1, \cdots, x_n)$ be the determinant of the matrix whose columns are the vectors $x_i=\{x_{i1}, \cdots, x_{in}\}$. Let $\Delta_r=(x_1, \cdots, x_{n-r}, \beta_1, \cdots, \beta_r)$, where $\beta_i$ is a column vector of constants. Let

$$\varphi=\Delta^{p_0}\Delta_1^{p_1}\cdots\Delta_{n-1}^{p_{n-1}}, \quad \varphi_1\Delta=\varphi,$$

where the $p_r$ are 0 or positive integers. Let

$$\lambda_r=p_0(p_0+p_1+1)\cdots(p_0+p_1+\cdots+p_{r-1}+r-1).$$

Let $\Omega=|\partial/\partial x_{ji}|$ be the Cayley operator. From an $n\times n$ matrix of arbitrary constants let the last $r$ columns be chosen and called $B$. Let

$$B_Z=(b_1b_2\cdots b_r|\partial/\partial z_1\,\partial/\partial z_2\cdots\partial/\partial z_r)$$

denote the compound inner product operator obtained by combining the $r$ columns of $B$ with the last $r$ columns of $\Omega$ (the $(n-r+1)$th set $x$ having been renamed $z_1$, etc.). Then

it is proved that $B_X\varphi = \lambda_r(X_{n-r}B)\varphi_1$. This generalizes the Cayley operator, the Capelli operator, and a main result in A. Young's substitutional analysis. *C. C. MacDuffee.*

**Souriau, Jean-Marie. Une méthode pour la décomposition spectrale et l'inversion des matrices.** C. R. Acad. Sci. Paris **227**, 1010–1011 (1948).

The author gives Leverrier's method for obtaining the characteristic equation $P(x)$ of the matrix $A$ in the following form. Let $B_0 = E$, $B_i = AB_{i-1} + k_iE$, with $k_i = -i^{-1}T(AB_{i-1})$, where $T$ is the trace. Then $P(x) = x^n + k_1x^{n-1} + \cdots + k_n$. Let, furthermore, $\lambda_i$ be a root of $P(x)$; then the corresponding eigencolumn and eigenrow can easily be calculated since Frobenius's covariant is $Q(\lambda_i)$, where

$$Q(x) = x^{n-1} + x^{n-2}B_1 + \cdots + B_{n-1}.$$

In the reviewer's opinion the author's method is not convenient for computing $\det A$ or $\operatorname{adj} A$, as the author proposes, since other methods are shorter. However, it has the advantage of employing only matrix calculations and therefore can be mechanized. On the other hand, the author's calculation of the eigenvectors seems interesting.
*E. Bodewig* (The Hague).

**San Juan Llosá, Ricardo. Exposition of some classical theorems of Galois theory.** Revista Acad. Ci. Madrid **42**, 71–78 (1948). (Spanish)

Assuming certain facts (for instance, about permutable substitution groups and about the relation of solvable groups to solvable equations), the author derives various known results about the group of the Lagrange resolvent, the Galois group of the binomial equation, imprimitive systems, and equations without affect. Metacyclic groups are a principal tool. *P. M. Whitman* (Silver Spring, Md.).

## *Abstract Algebra*

**Sebastião e Silva, José. Sugli automorfismi di un sistema matematico qualunque.** Pont. Acad. Sci. Comment. **9**, 327–357 (1945).

The author's summary (in Latin) is as follows. After defining the general concepts of a "mathematical system" and "automorphism" the author reaches conclusions which broaden as far as possible the doctrines expounded by F. Klein in his "Erlanger Programm" and thereby extend Galois theory to this domain. These conclusions not only systematize and clarify many things in all parts of mathematics, but can also be usefully applied to problems in functional analysis. The theorems which are announced here will shortly be demonstrated in another paper.
*K. A. Hirsch* (Newcastle-upon-Tyne).

**Matsushita, Shin-ichi. The algebra of topological operations. I.** Math. Japonicae **1**, 28–35 (1948).

A Kuratowski closure operator $\bar{A}$ can be used in a Boolean algebra to determine an abstraction of the topological theory in set theory. However the operation $\bar{A}$ alone does not suffice for such concepts as "dense-in-self" and "perfect" as used in set theory, nor for such theorems as the Cantor-Bendixson theorem, unless the Boolean algebra is atomistic. In a general, not necessarily atomistic, Boolean algebra, the author develops the entire theory from a postulated derivation operator $A^d$, the abstraction of the derived set. The operator $A^d$ is required to satisfy: $(A+B)^d = A^d + B^d$, $A^{dd} \leqq A^d$ and $0^d = 0$. [The author seems to be unaware that these conditions were given and studied by W. Sierpiński, Math. Ann. **97**, 321–337 (1926).] A number of identities are established, such as $A^{ddcd} = A^{dcd}$ and $A^{dcdcdcd} = A^{dcd}$, where $A^c$ denotes the complementation operator in the Boolean algebra.
*I. Halperin* (Kingston, Ont.).

**Inaba, Eizi. On primary lattices.** J. Fac. Sci. Hokkaido Univ. Ser. I. **11**, 39–107 (1948).

A "primary" lattice is defined as a modular lattice, whose quotients are all chains or sublattices with no neutral element; a "semi-primary" lattice, as a direct union (cardinal product) of "primary" lattices. The lattice of all subgroups of a finite Abelian group is semi-primary, being the direct union of the primary lattices of all subgroups of its Sylow subgroups. It is shown that many of the properties of subgroups of Abelian groups are valid in all "semi-primary" lattices: in particular, the theories of bases, invariants, and types can be extended; a primary lattice, not a chain, is simple; etc. [Reference might have been made to earlier results of R. Baer, Trans. Amer. Math. Soc. **52**, 283–343 (1942); these Rev. **4**, 109.] The deepest result obtained by the author is the realization of any primary lattice of rank $r \geqq 4$, by the lattice of all submodules in a group with operators over a "uniserial complete primary ring," in the sense of G. Köthe [Math. Z. **39**, 31–44 (1934)].
*G. Birkhoff* (Cambridge, Mass.).

**Balachandran, V. K. Ideals of the distribution lattice.** J. Indian Math. Soc. (N.S.) **12**, 49–56 (1948).

In a distributive lattice with 0 and 1, closed for product-complements [pseudo-complements; denoted by $'$], an element $x$ is called normal if $x \cap t = y \cap t$ and $t' = 0$ imply $x'' = y''$. If $x$ is a given normal element, then the set of elements with product-complement $x$ is characterized as the set of $x't$ with $t' = 0$. A normal ideal (ideal normal in the lattice of ideals) is the union of the principal ideals contained in it. The product-complement of an ideal $J$ equals the product-complement of the intersection of all principal ideals containing $J$. If $J$ is an ideal, $P$ is a given normal ideal, and $J' = P$, then $J = P' \cap K$ for some ideal $K$ such that $y \geqq k$ for all $k$ in $K$ implies $y' = 0$. If the lattice is complete, every normal ideal is principal. In the lattice of open sets of a topological space, "the most general ideal whose product-complement is a given normal ideal (principal ideal defined by an open domain $g$) is generated from an open covering of any open set whose exterior is $g$." *P. M. Whitman.*

**Schwan, W. Ein allgemeiner Mengenisomorphiesatz der Theorie der Verbände.** Math. Z. **51**, 346–354 (1948).

The author considers various properties of isotone transformations $S, T, \cdots$ of a lattice $L$. Sample result: the sets $ST$ and $TS$ of elements $x \varepsilon L$ which satisfy $xST = x$ and $xTS = x$, respectively, are isomorphic. *G. Birkhoff.*

**Pauc, Christian. Darstellungs- und Struktursätze für Boolesche Verbände und $\sigma$-Verbände.** Arch. Math. **1**, 29–41 (1948).

This is an exposition of the representation theory and structure theory of Boolean algebras and $\sigma$-algebras. The first section treats Stone's representation theorem from the point of view of ultrafilters, the second section takes up the related topological questions, and the third section deals with the representation of measure algebras and general $\sigma$-algebras. *L. H. Loomis* (Cambridge, Mass.).

**Kolchin, E. R. Existence theorems connected with the Picard-Vessiot theory of homogeneous linear ordinary differential equations.** Bull. Amer. Math. Soc. **54**, 927–932 (1948).

Let $I$ be a differential field over an algebraically closed field of constants $C$ of characteristic 0 and let $\Sigma$ be a nontrivial perfect differential ideal in the ring of differential polynomials in $n$ variables with coefficients in $I$. Then it is proved that $\Sigma$ has a solution $(\eta_1, \cdots, \eta_n)$ which does not involve any arbitrary constant (i.e., the field of constants of $I<\eta_1, \cdots, \eta_n>$ is still $C$). Moreover, if $J$ is a differential polynomial not in $\Sigma$, it may be assumed that $J(\eta_1, \cdots, \eta_n)\neq 0$. If $\Sigma$ has a solution in a differential field which is an extension of $I$ of one of the types considered by the author in a previous paper [Ann. of Math. (2) **49**, 1–42 (1948); these Rev. **9**, 561] but whose field of constants may be different from $C$, then $\Sigma$ has also a solution in an extension of the same type whose field of constants is $C$ (Liouvillian extension). It is also proved that, if a linearly irreducible homogeneous linear differential polynomial has a solution contained in a Liouvillian extension of a certain type, then it has a fundamental system of solutions in a Liouvillian extension of the same type. *C. Chevalley* (Paris).

**Leavitt, William, and Whaples, George. On matrices with elements in a principal ideal ring.** Bull. Amer. Math. Soc. **55**, 117–118 (1949).

By an algebraic approach, the authors obtain a simple proof of a theorem which is a generalization of a theorem of Leavitt [Duke Math. J. **15**, 463–472 (1948); these Rev. **10**, 6]. For the principal idea, cf. the cited review.
*L. K. Hua* (Urbana, Ill.).

**Szele, T. Eine kennzeichnende Eigenschaft der Schiefkörper.** Comment. Math. Helv. **22**, 115–116 (1949).

It is shown that in the following theorem the minimum condition for left ideals can be replaced by the existence of a minimum ideal: a ring without divisors of zero with minimum condition for left ideals is a field [cf. Artin, Nesbitt and Thrall, Rings with Minimum Condition, University of Michigan Press, Ann Arbor, Mich., 1944, p. 59; these Rev. **6**, 33]. *O. Todd-Taussky* (London).

**Foster, Alfred L. The $n$-ality theory of rings.** Proc. Nat. Acad. Sci. U. S. A. **35**, 31–38 (1949).

Partant d'un anneau $R$, l'auteur transforme les deux lois de composition de l'anneau par des permutations $\varphi$ de $R$ formant un groupe: pour chaque $\varphi$, une $\varphi$-addition est définie par $\varphi^{-1}(\varphi(a)+\varphi(b))$, une $\varphi$-multiplication par $\varphi^{-1}(\varphi(a)\varphi(b))$. Par exemple, si $R$ est un anneau booléien ayant un élément unité, $\varphi(x)=1-x$ transforme $ab=a\cap b$ en $a\cup b$. Pour un $p$-anneau $R$ (anneau tel que $pa=0$, $a^p=a$ identiquement, $p$ premier), l'auteur considère les $\varphi$ obtenus en itérant la permutation $x\to x+1$, et montre comment à toute proposition faisant intervenir les $\varphi$-additions et $\varphi$-multiplications on peut en faire correspondre $p-1$ autres, les "$p$-ales" de la première. Il discute les relations de ces notions avec la logique classique et ses généralisations (logiques à $n$ valeurs). *J. Dieudonné* (Nancy).

**Vinograde, B. Note on an invariant of commutative algebras.** Iowa State Coll. J. Sci. **23**, 101–102 (1948).

An algebra is cleft if it is the group direct sum of its radical and a semi-simple algebra; otherwise it is uncleft. The author proves that certain commutative algebras are uncleft; among these are some which were stated by Pickert [Math. Ann. **116**, 217–280 (1938)] to be cleft.
*N. H. McCoy* (Northampton, Mass.).

***Almeida Costa, A. Abelian groups, noncommutative rings and ideals, hyper-complex systems and representations. Vol. 2.** Centro Estudos Mat. Fac. Ci. Pôrto. Publ. no. **19**, 8+iv+518+xiii pp. (1948). (Portuguese)

The book under review is intended to be a continuation of the author's first volume [same Publ. no. **3** (1942); these Rev. **5**, 32]. In the first four chapters, after recalling the notions and basic properties of ideals, nil- and nilpotent ideals and radical in the ring case, the author proceeds to the study of rings which satisfy the ascending and descending conditions and the semi-primary rings. The next three chapters study linear (associative) algebras not necessarily of finite dimension and algebras of matrices, scalar extensions, simple algebras and Wedderburn theorems. The theory of representations of rings and algebras is then presented in two chapters followed by one dealing with group representation, a discussion of the rotation group of Euclidean space, the Lorentz group and spinors being also included. The two final chapters are devoted to the treatment of crossed products and cyclic algebras. Although, as declared by the author, this volume is in the main inspired by B. L. van der Waerden's "Moderne Algebra" [2d ed., Springer, Berlin, 1930] and by A. A. Albert's "Structure of Algebras" [Amer. Math. Soc. Colloquium Publ., v. 24, New York, 1939; these Rev. **1**, 99], it is not contained in these two books and as a matter of fact it presents more recent material as developed by R. Brauer, J. Dieudonné, N. Jacobson, J. Levitzky and the author. *L. Nachbin.*

**Albert, A. A. Power-associative rings.** Trans. Amer. Math. Soc. **64**, 552–593 (1948).

This paper embodies the results of extensive investigations by the author into the theory of non-associative rings, the investigations being linked by the fact that all rings considered are power-associative, i.e., such that the subrings generated by a single element are associative. The paper is divided up into 5 chapters, the headings being: I Nilrings and Idempotents; II Trace-admissible Algebras; III Commutative Shrinkable Algebras; IV Standard Algebras; V Quasi-associative Algebras.

Chapter I is mainly concerned with properties which are true either for all power-associative rings or, at least, for a wide class of such rings. The starting point of this theory is the theorem to the effect that a ring, either of characteristic zero or commutative and of characteristic prime to 30, in which the two identities $x^2\cdot x=x\cdot x^2$ and $x^2\cdot x^2=(x^2\cdot x)\cdot x$ are satisfied for all $x$, is power-associative. On this result, and several identities deduced from it, the author bases the theory of nil-ideals and idempotents in a power-associative ring. The main result obtained concerning nil-ideals is that the union of all nil-ideals in a power-associative ring is also a nil-ideal, which is termed the nil-radical. This, however, is not a satisfactory generalisation of the radical of an associative algebra, as, in general, it is not nilpotent or even solvable.

In the study of idempotents the author first restricts himself to commutative power-associative rings, and obtains a Pierce decomposition relative to an idempotent having many, but not all, of the properties of that obtained for Jordan algebras [Albert, Ann. of Math. (2) **48**, 546–567 (1947); these Rev. **9**, 77]. The decomposition is a direct decomposition into $A_e(1)+A_e(\frac{1}{2})+A_e(0)$, where $x\epsilon A_e(\lambda)$ if

$ex = \lambda x \cdot A_e(1)$, $A_e(0)$ are orthogonal sub-rings of $A$ and in addition the following inclusion relations hold:

$$A_e(\tfrac{1}{2})\cdot A_e(\tfrac{1}{2}) \subseteq A_e(1)+A_e(0),$$
$$A_e(1)\cdot A_e(\tfrac{1}{2}) \subseteq A_e(\tfrac{1}{2})+A_e(0),$$
$$A_e(0)\cdot A_e(\tfrac{1}{2}) \subseteq A_e(\tfrac{1}{2})+A_e(1).$$

In a non-commutative power-associative ring, the situation is more complicated, but if the equation $2x=a$ can always be solved uniquely, a decomposition theory can be obtained by replacing $A$ by the commutative ring $A^{(+)}$ in which the product is $x\cdot y = 2^{-1}(xy+yx)$. Then $A_e(1)$, $A_e(0)$ are still orthogonal with respect to the original multiplication, but are no longer sub-rings. If, however, we assume that the ring is flexible, i.e. the identity $x(yx)=(xy)x$ is satisfied, this decomposition has then most of the properties holding in the commutative case. If, further, the stronger inclusion relations, $A_e(\lambda)\cdot A_e(\frac{1}{2}) \subseteq A_e(\frac{1}{2})$, $A_e(\frac{1}{2})\cdot A_e(\lambda) \subseteq A_e(\frac{1}{2})$, hold for every idempotent, we have the further property that if $e$ is a principal idempotent, the elements of $A_e(\frac{1}{2})$ are nilpotent. Then $A$ is termed stable.

In the next chapter, the author restricts himself to algebras, and considers those stable algebras which admit a symmetric bilinear function $\tau(x, y)$ satisfying $\tau(xy, z) = \tau(x, yz)$, vanishing if $xy$ is nilpotent or zero, and such that $\tau(e, e) \neq 0$ if $e$ is an idempotent. Such an algebra is termed trace-admissible. The nil-radical of these algebras can be characterised as the set of all $x$ such that $\tau(x, y)=0$ for all $y$. Further, if $e$ is a principal idempotent, the nil-radical contains $A_e(\frac{1}{2})+A_e(0)$, leading to the result that a trace-admissible algebra without nilideals has an identity element.

In the third chapter, the author is concerned with the determination of commutative, shrinkable algebras [Albert, loc. cit.] of low level, which are power-associative. For level 1, this yields no new class of algebras, all such algebras being either associative or Jordan algebras of a special type. In the case of level 2, only those classes which consist entirely of power-associative algebras and which contain algebras of level 2 containing identity elements are considered. This yields only two classes of algebras, the first being that of Jordan algebras and the second is the class of static algebras, which are characterised by the identity $x^2\cdot y^2 \neq (xy)^2 = (x^2y)y+(y^2x)x$. A structure theory for static algebras is obtained. The static algebras without nil-ideals are all associative, and therefore direct sums of fields. The question of the existence of simple static nilalgebras, however, remains open.

In the fourth chapter, the author is mainly concerned with finding a class of algebras embracing both associative and Jordan algebras, to which the structure theorems common to both classes can be extended. He first considers the conditions on an algebra $A$ which imply that the algebras $A^{(+)}$ and $A^{(-)}$ (the latter having multiplication $x\cdot y = xy-yx$) should be, respectively, a Jordan algebra and a Lie algebra. He then defines, by means of two somewhat complicated identities, a class of algebras with both these properties which embraces both all associative algebras and all Jordan algebras. These algebras are termed standard algebras and are all flexible. In a standard algebra, the nil-radical is solvable and coincides with that of $A^{(+)}$. It follows that $A$ has no nil-ideal if and only if $A^{(+)}$ is a semi-simple Jordan algebra.

In the last chapter the theory of flexible $J$-semi-simple algebras, i.e. algebras for which $A^{(+)}$ is a semi-simple algebra, is developed. These are direct sums of $J$-simple algebras, so that the theory can be reduced to that of $J$-simple algebras. This latter theory depends on the introduction of quasi-associative algebras. The algebra $A(\lambda)$ is defined as the algebra with the same elements as $A$ but with multiplication $x\cdot y = \lambda xy + (1-\lambda)yx$. Then $B$ is termed quasi-associative if for some extension $K$ of the base-field there exists $\lambda$ in $K$ such that $B_K \cong A_K(\lambda)$, where $A$ is associative. These algebras are of shrinkability level 1 (unless $\lambda=\frac{1}{2}$, when they are Jordan algebras) and possess a structure theory similar to that of associative algebras. Their importance lies in the fact that all flexible $J$-simple algebras are either associative, quasi-associative, or Jordan algebras. This result can immediately be applied to standard algebras and yields the result that all standard simple algebras are either associative or Jordan algebras. *D. Rees.*

**Schafer, R. D. Structure of genetic algebras.** Amer. J. Math. **71**, 121–135 (1949).

The results of this paper consist of generalisations of results obtained by Etherington on certain nonassociative algebras occurring in the symbolism of genetics. The author defines a genetic algebra as follows. Let $A$ be a commutative algebra over a field $F$ with a homomorphism $x \to \omega(x)$ onto $F$. Then $\omega(x)$ is termed the weight of $x$. Let $T = \alpha I + F(R_{x_1}, R_{x_2}, \cdots)$ be an element of the transformation algebra of $A$. If the coefficients of the characteristic polynomial $|I-T|$ of $T$, in so far as they depend on the elements $x_i$, depend only on the weights $\omega(x_i)$, then $A$ is a genetic algebra. The author shows that genetic algebras occupy a position intermediate between train algebras [Etherington, Proc. Roy. Soc. Edinburgh **59**, 242–258 (1939); these Rev. **1**, 99] and commutative special train algebras [Etherington, loc. cit.; Quart. J. Math., Oxford Ser. **12**, 1–8 (1941); these Rev. **3**, 102]. Further, the duplicate of a genetic algebra [Etherington, Proc. Edinburgh Math. Soc. (2) **6**, 222–230 (1941); these Rev. **3**, 103] is also a genetic algebra. Next, the author shows that the kernel of the homomorphism $x \to \omega(x)$ is the radical of $A$ and is nilpotent. Finally genetic algebras which are also Jordan algebras are considered, the results obtained being somewhat sharper than the above. *D. Rees.*

**Dynkin, E. B. The structure of semi-simple algebras.** Uspehi Matem. Nauk (N.S.) **2**, no. 4(20), 59–127 (1947). (Russian)

This is an expository account of the structure theory of semi-simple Lie algebras over an algebraically closed field of characteristic 0. Virtually nothing is assumed, there being, in particular, an extensive introduction on vector spaces and linear transformations. The main body of the paper, summarized by sections, is as follows. (1) Fundamental concepts. Definitions and examples of Lie algebras. (2) Solvable and nilpotent algebras. The Lie-Engel theorems; invariant subspaces and reduction to triangular form. (3) Decomposition into regular nilpotent subalgebras (also called Cartan subalgebras). (4) Cartan's criterion for solvability and semi-simplicity. The expression of a semi-simple algebra as a direct sum of simple ones. (5–7) Systems of roots and the classification of simple algebras by root diagrams. An earlier account of this latter part has been given by the author [Rec. Math. [Mat. Sbornik] N.S. **18(60)**, 347–352 (1946); these Rev. **8**, 133].

*I. Kaplansky* (Chicago, Ill.).

**Ado, I. D. The representation of Lie algebras by matrices.** Uspehi Matem. Nauk (N.S.) **2**, no. 6(22), 159–173 (1947). (Russian)

Ado's theorem [Bull. Soc. Phys.-Math. Kazan (3) **7**, 3–43 (1936)] asserts that a Lie algebra over an algebraically

closed field of characteristic 0 can be faithfully represented by matrices. Other authors [Cartan, Malcev, Harish-Chandra, Hochschild] have since given new proofs (and dropped the hypothesis of algebraic closure). In the present paper the author gives an improved version of his earlier proof. The salient ideas are as follows. (1) A homomorphic image of a linear Lie algebra is again linear. This is proved by making use of the associated Birkhoff-Witt algebra. (2) Call $B$ a central extension of $A$ if $A \cong B/C$, where $C$ is one-dimensional and in the center. Then any nilpotent Lie algebra $L$ is shown to be a homomorphic image of the algebra of derivations of a suitable iterated central extension of $L$. At this point the theorem is known for the nilpotent case. (3) An elaborate lemma asserts that an algebra, obtained by modifying the multiplication table of a linear Lie algebra, is again linear. (4) The proof is completed by using the foregoing facts, plus the structure theory of Lie algebras, notably the Whitehead-Levi theorem.

*I. Kaplansky* (Chicago, Ill.).

# THEORY OF GROUPS

**Belova, E. N., Belov, N. V., and Šubnikov, A. V. On the number and character of the abstract groups corresponding to the 32 crystallographic classes.** Doklady Akad. Nauk SSSR (N.S.) **63**, 669–672 (1948). (Russian)

It is well known that the 230 space groups contain 32 finite subgroups, one for each of the crystal classes. The authors have tabulated [with at least three misprints] the operations of these finite groups in terms of the central inversion $C$, reflection $P$, rotation $L_n$, rotatory reflection $\bar{L}_{2n}$ and rotatory inversion $\dot{L}_{2n}$ (so that the symbols $\bar{L}_4$ and $\dot{L}_4$ are interchangeable). In this manner they show that certain sets of the groups are isomorphic, leaving eighteen distinct abstract groups: $C_1$, $C_2 \sim C_s \sim C_i$, $C_3$, $C_4 \sim S_4$, $V \sim C_{2h} \sim C_{2v}$, $C_6 \sim C_{3i} \sim C_{3h}$, $D_3 \sim C_{3v}$, $D_4 \sim C_{4v} \sim V_d$, $D_6 \sim C_{6v} \sim D_{3d} \sim D_{3h}$, $V_h$, $C_{4h}$, $D_{4h}$, $C_{6h}$, $D_{6h}$, $T$, $T_h$, $O \sim T_d$, $O_h$. They might well have gone on to observe that these eighteen are direct products of eight basic groups, as follows: $C_1$, $C_2$, $C_3$, $C_4$, $C_2 \times C_2$, $C_2 \times C_3$, $D_3$, $D_4$, $C_2 \times D_3$, $C_2 \times C_2 \times C_2$, $C_2 \times C_4$, $C_2 \times D_4$, $C_2 \times C_2 \times C_3$, $C_2 \times C_2 \times D_3$, $T$, $C_2 \times T$, $O$, $C_2 \times O$. The basic groups are: $C_n$, defined by $R^n=1$ ($n=1, 2, 3, 4$); $D_n$, defined by $R^n=S^2=(RS)^2=1$ ($n=3, 4$); $T$ and $O$, defined by $R^n=S^3=(RS)^2=1$ ($n=3, 4$).

*H. S. M. Coxeter.*

**Jaskowski, S. Sur l'application de la théorie générale de symétrie à la cristallographie.** Experientia **5**, 66–68 (1949).

The author proposes a new terminology for the 32 crystal classes, based on an alleged analogy between the trigonal axes of the cubic system and the principal axes of other systems. He calls Schoenflies's $C_n$ and $T$ rhythmic, $D_n$ and $O$ diametral, $C_{nv}$ and $T_d$ meridional, $C_{nh}$ equatorial, $C_{ni}$ and $T_h$ alternating, $D_{nh}$ crucial, $D_{nd}$ and $O_h$ undulating.

*H. S. M. Coxeter* (Toronto, Ont.).

**Nowacki, Werner. Symmetrie und physikalisch-chemische Eigenschaften kristallisierter Verbindungen. V. Über Ellipsenpackungen in der Kristallebene.** Schweiz. Mineral. Petrog. Mitt. **28**, 502–508 (2 plates) (1948).

Niggli has shewn [Z. Kristallogr., Mineral. Petrogr. Abt. A. **65**, 391–415 (1927); **68**, 404–466 (1928)] that there are 17 plane-groups and 31 packings of equal circles. The present author seeks the number of packings of equal ellipses. The symmetry elements of the ellipse can only be those of $C_{2v}-2m$, $C_s-m$, $C_2-2$ or $C_1-1$. The problem therefore is to find the number of ways in which the ellipse can be centred on the $C_{2v}$-, $C_s$-, $C_2$- and $C_1$-points of the 17 plane-groups so that it and its homologues form an ellipse-packing. Upon trial, 54 essentially different ellipse-packings have been found. In each of the cases $C_{2v}$, $C_s$ and $C_2$ the enumeration is complete, but in the case of $C_1$, owing to the lower symmetry, it can not be strictly proved that other packings do not exist. Drawings of each of the 54 packings are given, and their coordination number and plane-group along with that of the corresponding circle-packing (axes of ellipse equal) are tabulated.

*S. Melmore* (York).

**Piccard, Sophie. Les systèmes de substitutions qui engendrent le groupe symétrique ou le groupe alterné.** Ann. Univ. Lyon. Sect. A. (3) **11**, 21–29 (1948).

Given a connected set of $k>1$ cycles $S_1, \cdots, S_k$ of length $r$, affecting $n$ symbols in all, it is known that these cycles generate the symmetric group $\mathfrak{S}_n$ if $r=2$ and the alternating group $\mathfrak{A}_n$ if $r=3$. Taking $n>6$, the author proves that if the $S$'s form a primitive set then they also generate $\mathfrak{S}_n$ for $r=4$ and $\mathfrak{A}_n$ for $r=5$.

*G. de B. Robinson.*

**Kaloujnine, Léo, et Krasner, Marc. Le produit complet des groupes de permutations et le problème d'extension des groupes.** C. R. Acad. Sci. Paris **227**, 806–808 (1948).

Let $\Gamma_1, \cdots, \Gamma_s$ be groups of permutations of the sets $M_1, \cdots, M_s$ respectively, and let $M$ be the set of all $x$ of the form $(x_1, \cdots, x_s)$, where $x_i \epsilon M_i$, $i=1, \cdots, s$. Consider the mappings $\sigma$ of $M$ upon itself of the form $x \rightarrow \sigma x = y$, where $y=(y_1, \cdots, y_s)$, such that (i) $y_i$ depends only on $x_1, \cdots, x_i$ for $i=1, 2, \cdots, s$, and (ii) for fixed $x_1, \cdots, x_{i-1}$, the mapping $x_i \rightarrow y_i$ is a permutation of $M_i$ belonging to $\Gamma_i$. The set of all mappings $\sigma$ satisfying (i) and (ii) above is a group of permutations of the elements of $M$ which the authors call the "complete product" of $\Gamma_1, \cdots, \Gamma_s$. Several results are announced relative to the isomorphisms of an abstract group $H$ with a subgroup of such a complete product $G$, and it is stated that a new solution of the extension problem may be obtained from a consideration of the transitive subgroups of $G$ when $s=2$. A detailed discussion is to appear elsewhere.

*S. A. Jennings.*

**Taunt, D. R. On $A$-groups.** Proc. Cambridge Philos. Soc. **45**, 24–42 (1949).

A general construction theory for finite soluble groups has been given by P. Hall [J. Reine Angew. Math. **182**, 206–214 (1940); these Rev. **2**, 125] based upon the notion of "system normalisors" developed in earlier papers [see, for example, Proc. London Math. Soc. (2) **43**, 316–323, 507–528 (1937), where other references are given]. This theory is greatly simplified for soluble groups all of whose Sylow subgroups are Abelian, a fact which was noted by Hall who, in the first mentioned paper above, stated without proof certain properties of these groups. The present paper studies their properties in detail and in particular supplies proofs of all the results stated in Hall's paper.

Let $G$ be an $A$-group, i.e., a soluble group all of whose Sylow subgroups are Abelian. Since nilpotent $A$-groups are necessarily Abelian, the lower nilpotent series of $G$ coincides

with the derived series of $G$ (e.g., $G'$ is the smallest normal subgroup of $G$ whose factor group is nilpotent), while if $M$ is any system normalisor of $G$, $M$ is nilpotent and therefore Abelian. Perhaps the most important property of an $A$-group is the following: the intersection of the centre and the derived group in an $A$-group is the identity. The author draws many corollaries from this result, of which we mention only the following. The number of distinct Sylow systems of $G$ is equal to the order of $G'$; if $U$ is the maximal normal Abelian subgroup of $G$ (such a subgroup exists in any $A$-group) then $U=Z(G)\times Z(G')\times\cdots\times Z(G^{(n-1)})$, where $Z(G^{(i)})$ is the centre of the $i$th derived group of $G$. A basis theorem of the following type is proved. In an $A$-group $G$ there exists a system of permutable subgroups $G_1, G_2, \cdots, G_p$ whose product is $G$; each $G_i$ is Abelian of prime power order and type $(p_i^{\alpha_i}, p_i^{\alpha_i}, \cdots, p_i^{\alpha_i})$ and the product of those $G_i$ whose orders are powers of a prime $p$ dividing the order of $G$ is a Sylow $p$-subgroup of $G$.

Finally, the author studies groups of cube-free order. If such a group is soluble it is certainly an $A$-group, and it is known that all such groups of odd order are soluble, as are those of even order not containing a tetrahedral subgroup. It is shown that these groups of odd order are metabelian, while those of even order which are soluble have derived length at most 3. *S. A. Jennings* (Vancouver, B. C.).

**Baer, Reinhold. Finiteness properties of groups.** Duke Math. J. 15, 1021–1032 (1948).

Consider the following restrictions on a given group $G$: ($FO$) the group $G$ contains only a finite number of elements of any given order; ($LF$) every element of $G$ is contained in a finite normal subgroup of $G$; ($FC$) every element of $G$ has only a finite number of conjugates in $G$. The author shows that ($FO$) implies ($LF$) which implies ($FC$), but not conversely in either case. Clearly, all three hold if $G$ is of finite order but one or more may continue to hold in the contrary case. Groups which satisfy these conditions individually are characterised in three theorems, one of which we quote: $G$ is an $LF$-group if, and only if, (a) every element of $G$ has only a finite number of conjugates in $G$, and (b) $G$ does not contain elements of infinite order.

*G. de B. Robinson* (Toronto, Ont.).

**Zappa, Guido. Sui gruppi quasi-abeliani con elementi aperiodici.** Pont. Acad. Sci. Acta 6, 295–302 (1942).

**Zappa, Guido. Gruppi quasi-abeliani.** Pont. Acad. Sci. Acta 6, 249–267 (1942).

**Zappa, Guido. Caratterizzazione dei gruppi di Dedekind finiti.** Pont. Acad. Sci. Comment. 8, 443–460 (1944).

The structure of groups whose lattice of subgroups is modular ($M$-groups) was determined by K. Iwasawa, first for finite groups [J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. 4, 171–199 (1941); these Rev. 3, 193] and later for infinite groups [Jap. J. Math. 18, 709–728 (1943); these Rev. 7, 374]. The three papers now under review, together with one other not available to the reviewer but whose contents he has inferred from the others, constitute an independent, but in general less complete, attack upon the same problem. For convenience we shall refer to these papers as (1), (2) and (4), respectively, and to the missing paper as (3). A knowledge of Iwasawa's papers mentioned above is assumed in what follows.

In (1) and (2), and apparently in (3) also, the author is concerned with groups (necessarily modular) all of whose subgroups permute with each other [quasi-Abelian groups, called quasi-Hamiltonian in Iwasawa's work]. In (1) the structure of both finite and infinite quasi-Abelian groups with two generators is determined. The results obtained appear to be identical with those of Iwasawa, the ideas used being somewhat similar, although the details differ. In (2) finitely generated quasi-Abelian groups with elements of infinite order are discussed, and the theorem that such groups are either Abelian, or an extension of finite Abelian groups by an infinite cyclic group, is obtained. As far as the reviewer can gather from the brief reference in (4), the main results in (3) are that a finite quasi-Abelian group is the direct product of its (necessarily quasi-Abelian) Sylow subgroups and the determination of all such quasi-Abelian $p$-groups. In (4) the author is concerned with finite $M$-groups ["gruppi di Dedekind"]. He shows that any $M$-group which is not a direct product is either a quasi-Abelian $p$-group, and hence of the structure presumably determined in (3), or else is a modular $pq$-group of the same type as that given by Iwasawa. The general finite $M$-group is then the direct product of such groups. In a paper previously reviewed [Comment. Math. Helv. 18, 42–44 (1945); these Rev. 7, 113] the author gave a brief discussion of $M$-groups with elements of infinite order. [In a recent letter the author states that, after publishing the papers under review, he learned of Iwasawa's work and in consequence has not published the "missing paper" (3).] *S. A. Jennings.*

**Hua, Loo-Keng. On the automorphisms of the symplectic group over any field.** Ann. of Math. (2) 49, 739–759 (1948).

Soit $K$ un corps commutatif de caractéristique non 2, $E$ un espace vectoriel de dimension $2n$, $f(x, y)$ une forme alternée de rang $2n$ sur $E$; l'auteur détermine tous les automorphismes du groupe symplectique $Sp_{2n}(K)$ (formé des transformations linéaires $u$ telles que $f(u(x), u(y))=f(x, y)$). Il montre que ces automorphismes sont de la forme $u\rightarrow gug^{-1}$, où $g$ est une transformation semi-linéaire de $E$ (relative à un automorphisme quelconque $\sigma$ de $K$) telle que $f(g(x), g(y))=a\cdot f(x, y)$ identiquement. Pour $n=1$, $Sp_2(K)$ est identique au groupe unimodulaire, et le théorème résulte du théorème général de Schreier et van der Waerden [Abh. Math. Sem. Hamburg. Univ. 6, 303–322 (1928)] sur les automorphismes des groupes unimodulaires. L'auteur remarque d'ailleurs que la démonstration de Schreier et van der Waerden est incorrecte pour $n=1$, et indique une autre démonstration dans un appendice. Pour $n>1$, la détermination des automorphismes de $Sp_{2n}(K)$ est obtenue par récurrence sur $n$, l'hypothèse de récurrence étant appliquée au centralisateur d'une involution de type $(2, 2n-2)$ dans $Sp_{2n}(K)$ (c'est-à-dire une involution $u$ telle que $u(x)=x$ dans un sous-espace non isotrope de dimension 2, et $u(x)=-x$ dans le sous-espace conjugué); ce centralisateur est en effet isomorphe à $Sp_2(K)\times Sp_{2n-2}(K)$.

L'auteur emploie uniquement le langage matriciel pour exposer ses résultats; un langage plus géométrique permet de donner des démonstrations plus simples de certains de ces résultats, notamment du théorème 1 [cf. J. Dieudonné, Sur les groupes classiques, Actual. Sci. Ind., no. 1040=Publ. Inst. Math. Univ. Strasbourg (N.S.) no. 1 (1945), Hermann, Paris, 1948; ces Rev. 9, 494]. *J. Dieudonné.*

**Specht, W. Beiträge zur Darstellungstheorie der allgemeinen linearen Gruppe.** Math. Z. 51, 377–403 (1948).

L'auteur considère essentiellement le produit kroneckérien d'un certain nombre de représentations du groupe linéaire complexe, toutes identiques à la contragrédiente de la

représentation régulière. Suivant le procédé classique de Schur, il utilise la mesure de Haar du groupe unitaire pour obtenir d'abord les relations d'orthogonalité entre les caractères du groupe linéaire, puis les idempotents de l'algèbre de ce groupe. Il reprend ensuite la théorie bien connue de la relation entre représentations du groupe linéaire et représentations du groupe symétrique, et montre explicitement comment la donnée d'un idempotent de l'algèbre du groupe symétrique (ce qui revient à la donnée d'un idéal dans cette algèbre) détermine une représentation du groupe linéaire contenue dans le produit kroneckérien qu'il considère; il applique ce procédé à quelques idempotents particuliers, provenant de sous-groupes cycliques du groupe linéaire.

*J. Dieudonné* (Nancy).

**Croisot, Robert. Holomorphies d'un semi-groupe.** C. R. Acad. Sci. Paris **227**, 1134–1136 (1948).

Soit $D$ un semi-groupe associatif simplifiable des deux cotés. Le demi-groupe des applications biunivoques de $D$ dans $D$ a comme sous-groupoides $\Delta$ ensemble des applications de $D$ dans $D$ definies par $x \rightarrow sx$ et $G$ groupe des automorphismes de $D$. Soit $K$ la reunion de $\Delta$ et de $G$. Donc $K$ est l'holomorphie à gauche de $D$. C'est un semi-groupe simplifiable à droite et à gauche. L'auteur étudie l'intérieur et le centre de $K$.

*J. Kuntzmann* (Grenoble).

**Klein-Barmen, F. Ein Beitrag zur Theorie der linearen Holoide.** Math. Z. **51**, 355–366 (1948).

A "holoid" is a commutative groupoid in which $axy=a$ implies $ax=a$. Rather immediate consequences of this and related definitions are discussed, with simple illustrations.

*G. Birkhoff* (Cambridge, Mass.).

**Bates, Grace E., and Kiokemeister, Fred. A note on homomorphic mappings of quasigroups into multiplicative systems.** Bull. Amer. Math. Soc. **54**, 1180–1185 (1948).

If $G$ is a quasigroup and $G'$ is a homomorph of $G$, $G'$ is not necessarily a quasigroup; but up till now no example has been given of a quasigroup $G$ with an identity for which $G'$ is not a quasigroup. This paper gives (i) an example of this, (ii) a proof that $G'$ is a quasigroup if and only if $R(a\cdot b)=a\cdot R(b)=R(a)\cdot b$ for every $a$ and $b$ of $G$ (where $R(a)$ is the set of all $x$ for which $x'=a'$), and (iii) a proof that if $R(a)$ is a finite set, then $G'$ is a quasigroup. [In line 24, page 1181, $M$ is an error for $K$.]

*H. A. Thurston* (Bristol).

## NUMBER THEORY

**Zahlen, Jean-Pierre. Sur les sommes des chiffres successives d'un nombre.** Euclides, Madrid **8**, 260–265 (1948).

The author shows that if numbers are written in the base $b$ the sum of the digits of a number $N$ is given by

$$N-(b-1)\sum_{k=1}^{\infty}[Nb^{-k}],$$

where $[x]$ denotes the greatest integer not exceeding $x$. This result is applied to the discussion of the problem of Godey: given $N$ to find an integer $x$ such $N$ and $xN$ have the same sum of digits. Certain congruential conditions are found involving the digits of $x$. The problem is generalized to the case in which one iterates the procedure of summing the digits.

*D. H. Lehmer* (Berkeley, Calif.).

**Chowla, S. Proof of a theorem of Lerch and P. Kesava Menon.** Math. Student **15** (1947), 4 (1948).

A new proof is given of the result due to M. Lerch [Math. Ann. **60**, 471–490 (1905)] that $p^2$ is a divisor of $\sum_{a=1}^{p-1}a^{p-1}-p-(p-1)!$, where $p$ is any odd prime.

*I. Niven* (Eugene, Ore.).

**Clement, P. A. Congruences for sets of primes.** Amer. Math. Monthly **56**, 23–25 (1949).

Using Wilson's theorem, $((n-1)!+1\equiv 0 \pmod{n})$ if and only if $n$ is a prime), the author proves that $4[(n-1)!+1]+n\equiv 0 \pmod{n(n+2)}$ if and only if $n$ and $n+2$ are both primes. Similar results are also proved for prime triplets $n, n+2, n+6$; and prime quadruplets $n, n+2, n+6, n+8$. A method is proposed for obtaining this kind of result for prime sets of "any prescribed type."

*H. N. Shapiro* (New York, N. Y.).

**Pettineo, B. Sull'equazione indeterminata: $x^2+y^2=kz^n$.** Matematiche, Catania **1**, 180–210 (1946).

The author considers the Diophantine equation $x^2+y^2=kz^n$, and for any fixed $k$ and $n$ gives criteria for the existence of integral solutions. This is done by considering the factorization of both sides of the equation in the ring of Gaussian integers. In the special case $n=2$, well-known theorems of Gauss and Dirichlet are obtained.

*H. N. Shapiro.*

**Bell, E. T. The basic lemma in multiplicative Diophantine analysis.** Math. Gaz. **32**, 182–183 (1948).

The general solution of the Diophantine equation $xy=zw$ is $x=af$, $y=bk$, $z=ak$, $w=bf$, where $f$, $k$ may be taken relatively prime. It is proved that this result holds in any integral domain all of whose ideals are principal.

*I. Niven* (Eugene, Ore.).

**Bell, E. T. A Diophantine equation.** Amer. Math. Monthly **56**, 1–4 (1949).

Complete solutions of the Diophantine equation $x^2-y^2=z^3-w^3$ are given in terms of 9 independent parameters. The results are in two parts and are obtained by using complete solutions of the equations $rs=tv$, $r^2+3s^2=tv$ and $x_1y_1+x_2y_2+x_3y_3=0$.

*I. Niven* (Eugene, Ore.).

**Palamà, G. Tabelle della soluzione minima dell'equazione di Pell-Fermat.** Boll. Mat. (5) **2**, 40–42 (1948).

Bibliographical note.

**Racliş, Nicolas. Recherches sur le grand théorème de Fermat.** Ann. Roumaines Math. **5**, 61 pp. (1944).

This is a collection of the author's papers. Cf. these Rev. **7**, 47.

**Obláth, Richard. Note on the binomial coefficients.** J. London Math. Soc. **23**, 252–253 (1948).

P. Erdös [same J. **14**, 245–249 (1939); these Rev. **1**, 39] has shown that the binomial coefficient $\binom{n}{k}$ for $n\geqq 2k>2$ is not a perfect $j$th power if the equations $x^j\pm 1=2y^j$, $x^j\pm 1=2^{j-1}y^j$ have no solutions in positive integers (except $x=y=1$ for the first equation), a well-known fact for $j=3$. The writer points out that the impossibility of these equations has been proved for $j=4$ by Legendre, for $j=5$ by Dirichlet and V. A. Lebesgue.

*I. Niven* (Eugene, Ore.).

**Gentile, Giovanni. Considerazioni sui numeri perfetti dispari.** Period. Mat. (4) 26, 160–162 (1948).

**Erdös, P. On the number of terms of the square of a polynomial.** Nieuw Arch. Wiskunde (2) **23**, 63–65 (1949).

Let $Q(k)$ be the least number of terms occurring among the squares $f_k(x)^2$ of all polynomials $f_k(x)$, where $f_k(x)$ has rational coefficients and exactly $k$ nonzero terms. It is proved that there exist positive constants $c_1$, $c_2$ such that $c_1<1$ and $Q(k)<c_2k^{1-c_1}$. The proof uses the results [A. Rényi, Hungarica Acta Math. **1**, 30–34 (1947); these Rev. **9**, 182] that $Q(29)\leqq 28$ and $Q(ab)\leqq Q(a)Q(b)$, and is constructive when an $f_{29}(x)$ is known for which $f_{29}(x)^2$ has at most 28 terms. The conjecture of Rényi that $\lim_{k\to\infty} Q(k)=\infty$ is mentioned. *L. Tornheim* (Ann Arbor, Mich.).

**Borel, Émile. Sur une égalité numérique et sur l'addition vectorielle de certains ensembles.** C. R. Acad. Sci. Paris **227**, 1065–1066 (1948).

The author proves by using Farey series that $\sum 1/(qq')=\frac{1}{2}$ for any integer $n$, the summation being taken over all integer pairs $(q, q')$ mutually prime and such that $q\leqq n$, $q'\leqq n$, $q+q'>n$. *H. D. Ursell* (Leeds).

**Cohen, Eckford. An extension of Ramanujan's sum.** Duke Math. J. **16**, 85–90 (1949).

Let $c_q^s(n)=\sum \exp(2\pi i n h/q^s)$, where $n$ is a positive integer and $h$ runs over the nonnegative integers less than $q^s$ such that $h$ and $q^s$ have no common $s$th power divisors other than 1; $c_q^1(n)$ is the ordinary Ramanujan sum. Employing standard methods, the author first proves several theorems for $c_q^s(n)$ which correspond to known results for $c_q^1(n)$. He next derives similar theorems for an analogue of $c_q^s(n)$ in the ring of polynomials $GF[p^n, x]$. An application is given to the problem of finding the number of solutions of $F=\alpha_1X_1^sY_1+\cdots+\alpha_rX_r^sY_r$, where $F\epsilon GF[p^n, x]$, $\alpha_i\epsilon GF(p^n)$, and appropriate restrictions are made as to the degrees of $X_i$ and $Y_i$. Finally, a corresponding problem for the rational domain is discussed. *A. L. Whiteman.*

**Sengupta, H. M. On Ramanujan function $\tau(n)$.** Math. Student **15** (1947), 9–10 (1948).

If $p$ is a prime, the formula $\tau(p^n)=\tau(p)(p^{n-1})-p^{11}\tau(p^{n-2})$ is a well-known result conjectured by Ramanujan and proved by Mordell. This implies that $\tau(p^n)$ is a polynomial in $\tau(p)$ and $p^{11}$. The author finds this polynomial. It is the same as the polynomial which expresses $\csc x \sin nx$ in powers of $2\cos x$. *D. H. Lehmer* (Berkeley, Calif.).

**Gel'fond, A. O., and Linnik, Yu. V. On Thue's method in the problem of effectiveness in quadratic fields.** Doklady Akad. Nauk SSSR (N.S.) **61**, 773–776 (1948). (Russian)

Thue's method in the theory of Diophantine equations leads to upper bounds for the number of solutions, but in general not to bounds for these solutions themselves. The implications of this fact in the problem of obtaining all imaginary quadratic fields of class number 1 are discussed. Further, a generalized form of the Thue-Siegel theorem is given without proof; it gives the same exponent as Dyson's recent improvement of this theorem [Acta Math. **79**, 225–240 (1947); these Rev. **9**, 412]. *K. Mahler.*

**Koksma, J. F. On Niven's proof that $\pi$ is irrational.** Nieuw Arch. Wiskunde (2) **23**, 39 (1949).

A modification of an argument of Niven [Bull. Amer. Math. Soc. **53**, 509 (1947); these Rev. **9**, 10] is used to prove the irrationality of $e^s$ for any rational number $s\neq 0$, and thus also the irrationality of $\log r$ for positive rational values $r\neq 1$. The procedure is to assume $e^s=a/b$ for $s=p/q$, where $a$, $b$, $p$, $q$ are positive integers. Then for positive integral $n$ define $P_n(x)$ to be $bp^{2n+1}x^n(1-x)^n/n!$ so that $I_n=\int_0^1 P_n(x)dx$ is positive and tends to zero as $n$ increases indefinitely. A contradiction is obtained by repeated partial integration of $I_n$, which shows that $I_n$ is an integer, because $P_n(x)$ and all its derivatives are integers at $x=0$ and $x=1$. Minor correction: the last equation in the paper should have $e^s$ instead of $e^a$. *I. Niven* (Eugene, Ore.).

**Rado, R. An arithmetical property of the exponential function.** J. London Math. Soc. **23**, 267–271 (1948).

For $j=1, 2, \cdots, n$, let $f_j(x)$ be a set of real functions, not all identically zero, of the real variable $x$ satisfying the differential equations $f_j'=\sum_{k=1}^n c_{jk}f_k$ with determinant $|c_{jk}|\neq 0$. It is proved that for every rational number $\alpha$, with one possible exception, at least one of the numbers $f_j(\alpha)$ is irrational. Next let $P_j(x)$ be a set of $n$ polynomials with complex coefficients, $P_j(x)$ having degree less than an integer $m_j$. Let $w_j$ be a set of nonzero complex numbers such that $\prod_{j=1}^n(x-w_j)^{m_j}$ has rational coefficients, the degree of this product being say $r$. Define $F(x)=\sum_{j=1}^n P_j(x)\exp(w_jx)$. If $R\{F(x)\}$ is not identically zero, then for every rational number $\alpha$, with one possible exception, at least one of the numbers $F^{(q)}(\alpha)$ is irrational, $q=0, 1, 2, \cdots, r-1$. This second result is a corollary of the first, which is obtained by generalizing a method of the reviewer [Bull. Amer. Math. Soc. **53**, 509 (1947); these Rev. **9**, 10]. Two special cases of the second result are the irrationality of $e^b$ and $\tan b$ for any rational number $b\neq 0$. *I. Niven.*

**Fomin, A. M. On a class of nonlinear Diophantine approximations.** Doklady Akad. Nauk SSSR (N.S.) **63**, 7–10 (1948). (Russian)

Let $\phi(x)$ be a positive real function of the positive real variable $x$, with $x\phi''(x)$ a nondecreasing function of $x$, and $\phi''(x)$ a monotone decreasing function tending to zero as $x\to\infty$. Let $r$ be a given integer, $N$ any sufficiently large number, and $\mu$ the solution of the equation $N=r\phi(\mu)$. Then there exist integers $x_1, \cdots, x_r$, satisfying the inequalities

$$|x_i-\mu|<6r^{\frac{1}{2}}(\phi'(\mu)/\phi''(\mu))^{\frac{1}{2}}, \qquad i=1, \cdots, r,$$
$$0<\phi(x_1)+\cdots+\phi(x_r)-N<c_r(\phi'(\mu)/\phi''(\mu))^{2^{1-r}}\phi''(\mu),$$

with $c_r$ a constant depending only on $r$. The author states the above theorem, with two others of a similar but more general character. Proofs are only briefly sketched. *F. J. Dyson* (Princeton, N. J.).

**Teghem, J. Sur un système d'inéquations diophantiennes.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 593–603 (1948).

The following theorem is proved. Let $\delta<1$, let $c_1$ and $c_2$ be positive constants, and let $\theta_1, \theta_2, \cdots$ be a set of real numbers with $\theta_1>\frac{4}{3}$ and $\theta_\nu-\theta_{\nu-1}\geqq c_1\nu^{-1}$ $(\nu=2, 3, \cdots)$. Let $n$ be a function of $x$ going to infinity with $x$, and let $\phi(n)$ be a positive increasing number-theoretic function such that $\lim(\log\phi(n)/\log n)>0$ and $\liminf(\log\phi(n)/\log\theta_n)\geqq 3$ as $n\to\infty$. Let $c_2\exp(-\phi(\nu))\leqq\theta_\nu-[\theta_\nu]\leqq 1-c_2\exp(-\phi(\nu))$ for $\nu=1, 2, \cdots$. Then the system of inequalities

$$0<x^{\theta_\nu}<\exp(-\log^\delta x) \pmod 1,$$

$(\nu=1, 2, \cdots, n)$ has infinitely many integral solutions $x$ if $\limsup(\log\phi(n)+\log n)/\log\log x<1-\delta$ as $x\to\infty$. This generalizes an earlier theorem of J. F. Koksma [thesis, Groningen, 1930] in which $\phi(n)=2^{c_3n}$ and $\theta_\nu=c_3\nu+\omega_\nu$,

where $c_3>0$ and $\omega_1, \omega_2, \cdots$ are bounded and such that $c_3+\omega_\nu-\omega_{\nu-1}\geqq\alpha>0$ for some $\alpha$. The proof makes use of an estimate of Weyl sums obtained by the author in a paper which has apparently not yet appeared.

*W. J. LeVeque* (Cambridge, Mass.).

**Gál, I. S. A theorem concerning Diophantine approximations.** Nieuw Arch. Wiskunde (2) **23**, 13–38 (1949).

Let $[a, b]$ be the ratio of the g.c.d. and the l.c.m. of $a$ and $b$, so that if $a=\prod p^\alpha$, $b=\prod p^\beta$, then $[a, b]=\prod p^{-|\alpha-\beta|}\leqq 1$. It is shown that if $n_1, \cdots, n_N$ $(N>N_0)$ is any set of positive integers, there are constants $C$, $c$ such that always $\sum_{i,j\leqq N}[n_i, n_j]<CN(\log\log N)^2$, and for a suitable set of $N$ integers this sum is larger than $cN(\log\log N)^2$. In the case where the $n_i$ are restricted to be square-free, these bounds can be replaced by $C'N\log\log N$ and $c'N\log\log N$, respectively. The proofs are elementary but quite complicated. It is first shown that the above sum will not be maximal unless every divisor of each $n_i$ is also an $n_i$; from this it follows that if the sum is maximal then the corresponding $n_i$ are products of powers of at most the first $c_1\log N$ primes. This implies that $\sum_{j=1}^n[n_u, n_j]<c_2\log\log N$ for suitably chosen $u$, and the statements follow by induction on $N$. As an application, the author uses the identity $\int_0^1\{ax\}\{bx\}dx=[a, b]/12$, where $\{ax\}=ax-[ax]-\frac{1}{2}$, together with his main theorem, to show that for every sequence $n_1<n_2<\cdots$ of positive integers, $\sum_{i=1}^n\{n_i x\}=o(N^{\frac{1}{2}}\log^{2+\epsilon}N)$ for almost all $x$ and any $\epsilon>0$, and states that a minor modification in the proof will yield the better bound $o(N^{\frac{1}{2}}\log^{\frac{3}{2}+\epsilon}N)$.

*W. J. LeVeque* (Cambridge, Mass.).

**Sawyer, D. B. The product of two non-homogeneous linear forms.** J. London Math. Soc. **23**, 250–251 (1948).

Let $x=a\xi+b\eta+p$, $y=c\xi+d\eta+q$ define a nonhomogeneous lattice of determinant $\Delta=|ad-bc|$, where $a, b, c, d, p, q$ are real. Minkowski showed that integral values of $\xi$, $\eta$ exist such that (1) $|xy|\leqq\Delta/4$. The author finds another proof by considering lattice points in relation to the region $|xy|\leqq\Delta/4$. This short proof may also be used to determine those lattices for which equality in (1) is necessary.

*L. Tornheim* (Ann Arbor, Mich.).

**Mullineux, N. On two problems of K. Mahler on irreducible star domains.** Ann. Mat. Pura Appl. (4) **26**, 375–382 (1947).

To answer two problems proposed by Mahler [Nederl. Akad. Wetensch., Proc. **49**, 444–454 = Indagationes Math. **8**, 299–309 (1946), problems 3 and 4, pp. 449–450; these Rev. **8**, 12] the author constructs plane bounded star bodies $K_d$, depending on a real parameter $d$ (the boundary curves are either hyperbolas or straight lines), which are irreducible, i.e., $\Delta(K)<\Delta(K_d)$ if $K<K_d$, and which have the following two properties. (1) Boundary points $X$, $X_1$ and $X_2$ of $K_d$ exist so that the origin, $X$ and $X_1+X_2$ are collinear and $|X_1+X_2|/|X|$ becomes infinite with $d$; (2) $V(K_d)/\Delta(K_d)$ becomes infinite with $d$, where $V(K_d)$ is the area of $K_d$.

*D. Derry* (Vancouver, B. C.).

**Mahler, K. On the critical lattices of arbitrary point sets.** Canadian J. Math. **1**, 78–87 (1949).

Let $S$ be any point set within Euclidean $n$-space. A lattice is defined to be $S$-admissible if its only point interior to $S$ is the origin. Let $\Delta(S)$ be the lower bound of the determinants of all the $S$-admissible lattices. If $0<\Delta(S)<\infty$, a lattice with determinant $\Delta(S)$ is defined to be critical. The author shows easily that $S$ has a critical lattice if a sphere with center at the origin exists together with an infinite sequence of $S$-admissible lattices whose determinants are bounded and none of which have points other than the origin in the sphere. The main part of the paper deals with the construction of a set $S$ which has admissible but no critical lattices. This set is the part of Euclidean $n$-space whose coordinates exceed 1 after a certain infinite set of open parallelepipeds has been removed. All the $S$-admissible lattices are shown to be generated by centers of the parallelepipeds not in $S$. Their determinants are shown to be greater than 1 while $\Delta(S)=1$. Hence no critical lattices can exist. This system of $S$-admissible lattices is enumerable. Another example of a space $S$ is given whose only admissible lattices are the lattice of all points with integral coordinates and its sublattices. This space consists of all points $(x_1, \cdots, x_n)$ for which $\max[|x_1-u_1|, \cdots, |x_n-u_n|]\geqq\frac{1}{2}$, where $(u_1, \cdots, u_n)$ runs through all vectors with integral components.

*D. Derry* (Vancouver, B. C.).

***Berker, Ratip. Asal Sayılar. [Prime Numbers].** Üniversite Kitabevi, Istanbul, 1948. viii+47 pp. (1 plate).

This work gives an up-to-date exposition for Turkish students of the theory of primes from Euclid's theorem to the prime number theorem. Proofs are given of the elementary theorems, the more difficult ones being illustrated and discussed historically. There is a complete proof of the theorem of Chebyshev. The first page of Lehmer's list of primes is reproduced as well as the Jahnke-Emde relief of the zeta function.

*D. H. Lehmer* (Berkeley, Calif.).

**Jarden [Juzuk], Dov. On the appearance of prime factors in the sequence associated with Fibonacci's sequence.** Math. Student **15** (1947), 11–12 (1948).

Let $u_0=0$, $u_1=1$, $u_{n+1}=u_n+u_{n-1}$ be the Fibonacci sequence and let $v_k=u_{2k}/u_k$. Then the odd primes $p$ fall into two classes: $p$ belongs to class 1 in case $p$ divides $v_k$ for some $k$, otherwise $p$ belongs to class 2. All primes congruent to 3, 7, 11, 19 (mod 20) are known to belong to class 1 and all $p$ congruent to 13 and 17 (mod 20) belong to class 2. This leaves in doubt the cases $p\equiv 1, 9$ (mod 20). It is shown in this note that infinitely many primes $p\equiv 1$ (mod 20) belong to class 1, and infinitely many to class 2. The author poses the question: do those primes $20n+1, 9$ which belong to class 1 belong to a set of arithmetical progressions?

*D. H. Lehmer* (Berkeley, Calif.).

**Čulanovskiĭ, I. V. Certain estimates connected with a new method of Selberg in elementary number theory.** Doklady Akad. Nauk SSSR (N.S.) **63**, 491–494 (1948). (Russian)

The author indicates how a result of Selberg [Norske Vid. Selsk. Forh., Trondhjem **19**, no. 18, 64–67 (1947); these Rev. **9**, 271] can be applied to give elementary proofs of the four theorems stated below. These results are somewhat sharper than those obtained by Brun's method.

(I) Let $\pi(x; k, l)$ denote the number of prime numbers not exceeding $x$ which lie in the arithmetic progression $kn+l$, where $(k, l)=1$. Suppose that $k=O(x^\delta)$, where $\delta$ is a fixed number between 0 and 1. Then

$$\pi(x; k, l)\leqq\frac{2x}{\phi(k)\ln(x/k)}\left\{1+O\left(\frac{\ln\ln x}{\ln x}\right)\right\},$$

where $\phi$ is Euler's function.

(II) Let $Z(u; N)$ denote the number of positive integers $n$ not exceeding $N$ such that $n$ and $n+u$ are both primes.

where $u$ is an even natural number possibly varying with $N$. Then

$$Z(u;N)\leqq 16\prod_{2<p}\frac{p(p-2)}{(p-1)^2}\times\prod_{2<p|u}\frac{p-1}{p-2}\frac{N}{(\ln N)^2}\left\{1+O\left(\frac{\ln\ln N}{\ln N}\right)+O\left(\frac{(\ln u)^3}{(\ln N)^2(\ln\ln N)^2}\right)\right\}.$$

(III) Let $0, u_1, \cdots, u_{m-1}$ be distinct fixed nonnegative integers not forming a complete system of residues with respect to any prime modulus. Let $Z(u_1, \cdots, u_{m-1}; N)$ denote the number of positive integers $n$ not exceeding $N$ such that $n, n+u_1, \cdots, n+u_{m-1}$ are all primes. Then

$$Z(u_1,\cdots,u_{m-1};N)\leqq\frac{2^m m!N}{(\ln N)^m}\prod_p\frac{1-\omega(p)/p}{(1-1/p)^m}\{1+o(1)\},$$

where $\omega(p)$ denotes the number of distinct residue classes modulo $p$ represented by $0, u_1, \cdots, u_{m-1}$.

(IV) If $A(N)$ denotes the number of representations of an even positive integer $N$ as a sum of two primes, then

$$A(N)\leqq 16\prod_{2<p}\frac{p(p-2)}{(p-1)^2}\prod_{2<p|N}\frac{p-1}{p-2}\cdot\frac{N}{(\ln N)^2}\left\{1+O\left(\frac{1}{(\ln\ln N)^2}\right)\right\}.$$

The first of these theorems is proved by using Selberg's method to estimate the number of positive integers not exceeding $x$ which lie in the arithmetic progression $kn+l$ and which are not multiples of any of the primes not dividing $k$ and not exceeding $N^{\frac{1}{2}}/(\ln N)^2$, where $N=[(x-l)/k]+1$. The other theorems are proved similarly.

*P. T. Bateman* (Princeton, N. J.).

**Lukomskaya, M. A. A new proof of the theorem of van der Waerden on arithmetic progressions and some generalizations of this theorem.** Uspehi Matem. Nauk (N.S.) 3, no. 6(28), 201–204 (1948). (Russian)

The theorem in question asserts that for any natural numbers $k, l$ there is a number $n=n(k, l)$ such that if the numbers $1, 2, \cdots, n$ are split up in any way into $k$ classes, then one at least of these classes contains an arithmetical progression of length $l$. The author's generalization is: given natural numbers $t_1, \cdots, t_{l-1}$, and given $k$, there is a number $n$ such that one at least of the classes contains numbers $c_1, \cdots, c_l$ such that $c_2-c_1:c_3-c_2:\cdots:c_l-c_{l-1}=t_1:t_2:\cdots:t_{l-1}$. In fact, however, this is a trivial generalization, since any sufficiently long arithmetical progression will contain $l$ numbers which satisfy the required condition. We have only to pick them out from the arithmetical progression in the obvious way. The proof is also not new, being essentially the same as van der Waerden's original proof [Nieuw Arch. Wiskunde (2) 15, 212–216 (1927)]. *H. Davenport.*

**Borchsenius, Vibeke, and Jessen, Børge. Mean motions and values of the Riemann zeta function.** Acta Math. 80, 97–166 (1948).

The authors employ the notion of generalized almost periodic functions in a half-strip in studying the functions $\log\zeta(s)-x$ and $\zeta(s)-x$ in the half-plane $\sigma>\frac{1}{2}$. These functions are shown to have mean motions and twice differentiable Jensen functions. Very complete relationships are found between the Jensen functions, mean motions, asymptotic distributions and relative frequency of zeros for these functions. *R. H. Cameron* (Minneapolis, Minn.).

**Estermann, T. On Dirichlet's $L$ functions.** J. London Math. Soc. 23, 275–279 (1948).

Here there is given a complete proof of Siegel's theorem that, if $\chi$ is a real primitive character mod $k$, then, corresponding to each $\epsilon>0$, there is an $A(\epsilon)>0$ such that $L(1,\chi)>A(\epsilon)/k^\epsilon$. This proof demands less previous knowledge than both the original proof of Siegel [Acta Arith. 1, 83–86 (1935)] and a later proof of Heilbronn [Quart. J. Math., Oxford Ser. 9, 194–195 (1938)]. The only special knowledge required is the product representation of $L(s,\chi)$, the fact that $L(s,\chi)$ is regular for $\sigma>0$ if $\chi$ is a real primitive character mod $k$, $k>1$, $\sum_{n=1}^{k}\chi(n)=0$ for the same type of $\chi$, $L(1,\chi)\neq 0$, and that $\zeta(s)-1/(s-1)$ is regular for $\sigma>0$. In addition, Cauchy's inequality for the coefficients of a Taylor expansion is used. *L. Schoenfeld* (Urbana, Ill.).

**Fine, Nathan J. Some new results on partitions.** Proc. Nat. Acad. Sci. U. S. A. 34, 616–618 (1948).

Five theorems on restricted partitions are announced. Proofs are promised in a later paper. The theorems are as follows. (1) The number of partitions of $n$ into distinct parts, the smallest part being odd, is odd if and only if $n$ is a square. (2) The number of partitions of $n$ into odd parts of which the largest is $r$ is equal to $D_r(n)+D_{r-1}(n)$, where $D_s(n)$ denotes the number of partitions of $n$ into distinct parts in which the maximum part exceeds the number of parts by $s$. (3) The difference between the numbers of unrestricted partitions of $n$ and $n-1$ is equal to the number of partitions of $r+n+1$ in which the largest part exceeds the number of parts by $r$, where $r$ is a positive integer exceeding $n-4$. Denoting the latter number of partitions by $P_r(r+n+1)$ we have (4) $P_r(n)+P_{r+4}(n-r-3)=P_r(n-1)+P_{r+3}(n-r-2)$. (5) Let $E(n)$ denote the excess of the number of partitions of $n$ into distinct parts in which the largest part is even over the number of partitions into distinct parts the largest part being odd. Then $E(n)=0$ unless $n$ is pentagonal in which case $E(\frac{1}{2}(3k^2+k))=1$ or $-1$ according as $k\geqq 0$ or not. This result parallels a famous theorem of Euler.

$P_{r+1}(n)$

*D. H. Lehmer* (Berkeley, Calif.).

**Srinivasan, A. K. Practical numbers.** Current Sci. 17, 179–180 (1948).

By a "practical" number is meant an integer $N$ such that every integer not exceeding $N$ is a sum of distinct parts taken from the set of divisors of $N$. Thus 12, but not 10, is a practical number. The even perfect numbers are practical as well as the products of the first $n$ primes. There are 49 practical numbers not exceeding 200; these fall into three classes which are described in general. One of these includes the "highly composite" numbers of Ramanujan.

*D. H. Lehmer* (Berkeley, Calif.).

**Cugiani, M. Osservazioni relative alla questione dell' esistenza di un algoritmo euclideo nei campi quadratici.** Boll. Un. Mat. Ital. (3) 3, 136–141 (1948).

The author proves the following lemma (all symbols denote positive integers). Suppose that $p$ is a prime, $p\equiv h \pmod{k}$, that $k'|k$, $h'<k'$, $d=(k', h-h')$, $h'\equiv(h-h')/d \pmod{k'/d}$, $d\geqq t$, $p>dtk'$, and that $-d$ is a quadratic nonresidue of $p$. Then there exists a number congruent to $h' \pmod{k'}$ and less than $p/t$ which is a quadratic nonresidue of $p$. He applies this lemma to fill an alleged gap in Hofreiter's proof [Monatsh. Math. Phys. 42, 397–400

(1935)] that the Euclidean algorithm does not hold in $k(\sqrt{D})$ if $D\equiv 21 \pmod{24}$ and $D>21$. He also uses it to prove the same if $D\equiv 13$ or $27 \pmod{120}$ and $D$ is a prime greater than $15^3$. *H. Davenport* (London).

**Lehner, Joseph. Divisibility properties of the Fourier coefficients of the modular invariant $j(\tau)$.** Amer. J. Math. 71, 136–148 (1949).

If $c_\nu$, $\nu>1$, are the coefficients of the modular invariant $j(\tau)=x^{-1}+744+\sum_{\nu=1}^{\infty}c_\nu x^\nu$, $x=e^{2\pi i\tau}$, then $c_\nu\equiv 0 \pmod{5^{\alpha+1}}$ if $\nu\equiv 0 \pmod{5^\alpha}$, $\alpha=1, 2, 3, \cdots$; $c_\nu\equiv 0 \pmod{7^\alpha}$ if $\nu\equiv 0 \pmod{7^\alpha}$, $\alpha=1, 2, 3, \cdots$; $c_\nu\equiv 0 \pmod{11^\alpha}$ if $\nu\equiv 0 \pmod{11^\alpha}$, $\alpha=1, 2, 3$. The proof depends on subjecting $j(\tau)$ to a certain linear operator and expressing the result in terms of basic modular functions. The identities thus obtained lead directly to the congruence properties. The method is that previously used by Rademacher [Trans. Amer. Math. Soc. 51, 609–636 (1942); these Rev. 3, 271] and the author [same J. 65, 492–520 (1943); these Rev. 5, 34] to obtain identities involving the partition function. Besides 5, 7, 11 the prime 13 also leads to an identity but not to congruence properties of the $c_\nu$. *H. S. Zuckerman* (Seattle, Wash.).

**Lehner, Joseph. Further congruence properties of the Fourier coefficients of the modular invariant $j(\tau)$.** Amer. J. Math. 71, 373–386 (1949).

This is an extension of the paper reviewed above. Using the same notation, it is shown that $c_\nu\equiv 0 \pmod{2^{3\alpha+8}}$ if $\nu\equiv 0 \pmod{2^\alpha}$ and $c_\nu\equiv 0 \pmod{3^{2\alpha+3}}$ if $\nu\equiv 0 \pmod{3^\alpha}$; $\alpha, \nu=1, 2, 3, \cdots$. *H. S. Zuckerman* (Seattle, Wash).

**Sominskiĭ, I. S. On the existence of automorphisms of the second order for certain ternary quadratic indefinite forms.** Mat. Sbornik N.S. 23(65), 279–296 (1948). (Russian)

Let $f(x, y, z)=ax^2+by^2+cz^2+2gyz+2hzx+2kxy$ be an indefinite form with integer coefficients. The author attacks the problem of the existence of involutory automorphs of $f$. He obtains some partial results. (1) Every form where invariants are odd relatively prime numbers, possesses involutory automorphs; (2) every nonzero form whose determinant is a power of an odd prime possesses involutory automorphs.

It should be remarked that, on p. 284, he proves that a form which has an involutory automorph can be transformed by means of an unimodular integral substitution into one of the following:

$$ax^2+by^2+cz^2+2gyz, \tag{1}$$
$$2a_1x^2+by^2+cz^2+2a_1xy+2gyz, \tag{2}$$
$$2a_1x^2+by^2+cz^2+2a_1xy+2a_1xz+2gyz; \tag{3}$$

but (2) and (3) are equivalent. In fact,

$$\begin{aligned}2a_1x^2+by^2+cz^2+2a_1xy+2a_1xz+2gyz&=2a_1(x+z)^2\\ +b(y-z)^2+(-2a_1+b+c+2g)z^2&+2a_1(x+z)(y-z)\\ &+2(b+g-a_1)(y-z)z.\end{aligned}$$

*L. K. Hua* (Urbana, Ill.).

# ANALYSIS

**Aczél, Jean. Un problème de M. L. Fejér sur la construction de Leibniz.** Bull. Sci. Math. (2) 72, 39–45 (1948).

Leibniz [Analysis des Unendlichen, translated in Ostwald's Klassiker der exakten Wissenschaften, v. 162, Leipzig, 1908, pp. 12–18] constructed the exponential curve $y=a^x$ as solution of the problem of determining a curve $y=f(x)$ such that the geometric mean of ordinates corresponds to the arithmetic mean of abscissas: $f(\frac{1}{2}(x_1+x_2))=[f(x_1)f(x_2)]^{\frac{1}{2}}$. In response to a suggestion of Fejér, the author investigates solutions of (1) $f(q_1x_1+q_2x_2)=M[f(x_1), f(x_2); q_1, q_2]$ for certain means $M$ with weights $q_j>0$, $q_1+q_2=1$; he shows that if the mean $M(u, v; q_1, q_2)$ is reflexive, homogeneous, and bisymmetric, then the function $f(x)=M(a, b; 1-x, x)$, for arbitrary $a$, $b$, satisfies (1). *E. F. Beckenbach.*

**Sibirani, Filippo. Questioni di massimo e di minimo.** Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 4 (1946–47), 111–117 (1948).

The proposition, established in this note subject to unnecessary restrictions, is actually an immediate consequence of the definition of rearrangement [Hardy, Littlewood and Pólya, Inequalities, Cambridge University Press, 1934, 10.12.2] and of the fact that the exponential function is increasing. *L. C. Young* (Madison, Wis.).

**Tagamlitzki, Y. Sur une propriété de la fonction exponentielle.** C. R. Acad. Bulgare Sci. Math. Nat. 1, 33–34 (1948).

The author gives an elementary proof of his theorem that an infinitely differentiable function $f(x)$ such that $|f^{(n)}(x)|\leqq Ae^{-x}$, $n=0, 1, 2, \cdots$; $x\geqq a$ is necessarily of the form $f(x)=Ce^{-x}$. *H. Pollard* (Ithaca, N. Y.).

**Rollero, Aldo. Sul calcolo grafico di alcuni limiti notevoli.** Atti Accad. Ligure 4 (1947), 32–42 (1948).

La fonction $\varphi(x)$ étant définie dans un intervalle $(a, b)$, continue ainsi que sa dérivée première $\varphi'(x)$ et $x_0$ étant un point de $(a, b)$, l'auteur cherche graphiquement des critères de convergence de la suite des itérées $x_1=\varphi(x_0)$, $x_2=\varphi(x_1)$, $\cdots$. Il s'appuie sur ce que, s'il y a convergence vers un nombre fini $\lambda$, ce nombre est l'abscisse d'un des points communs $S$ à la droite $y=x$ et à la courbe définie par $y=\varphi(x)$. Il étudie la position relative de cette courbe et de cette droite dans le voisinage de $S$, en faisant les hypothèses classiques sur la valeur de $\varphi'(\lambda)$. *G. Valiron* (Paris).

**Obrechkoff, N. Sur quelques inégalités pour les dérivées des fonctions et les différences des suites.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 5, 21–24 (1948).

**Obrechkoff, N. Sur quelques inégalités pour les dérivées et les différences des fonctions d'une variable réelle et pour les différences des suites.** C. R. Acad. Bulgare Sci. Math. Nat. 1, no. 1, 1–4 (1948).

The author gives further inequalities of the same character as in other recent notes [C. R. Acad. Sci. Paris 224, 880–882 (1947); Doklady Akad. Nauk SSSR (N.S.) 59, 1399–1401 (1948); these Rev. 8, 448, 709; 9, 416]. *R. P. Boas, Jr.* (Providence, R. I.).

**Aljančić, S. Sur une formule sommatoire généralisée.** Acad. Serbe Sci. Publ. Inst. Math. 2, 263–269 (1948). (French. Serbian summary)

Let $\alpha(x)$ be of bounded variation in $(a, b)$, with jumps at $x_\nu$ and continuous part $\bar{\alpha}(x)$; let $f(x)$ have an $n$th derivative in $(a, b)$. By integration by parts one obtains the formula

[attributed to Karamata]

$$\sum\{\alpha(x_\nu+)-\alpha(x_\nu-)\}f(x_\nu)+\int_a^b f(x)d\bar{\alpha}(x)$$
$$=\sum_{k=0}^{n-1}(-1)^k\{\alpha_k(b)f^{(k)}(b)-\alpha_k(a)f^{(k)}(a)\}$$
$$+(-1)^n\int_a^b f^{(n)}(x)d\alpha_n(x),$$

where $\alpha_k(x)$ is a $k$th integral of $\alpha(x)$+constant. By proper choice of the constants of integration one obtains both Taylor's formula and the Euler-Maclaurin formula; for a generalized form of the latter take $\alpha_k(b)=\alpha_k(a)$; the classical case results when $a=0$, $b=N$, $\alpha(x)=x-[x]$. The object of the paper is to point out that the generalized Euler-Maclaurin formula, like the classical one, generally leads only to asymptotic (not convergent) series when $n\rightarrow\infty$. [It should be remarked that Karamata's formula is practically a special case of one due to Kronecker, S.-B. Preuss. Akad. Wiss. Berlin **1885**, 841–862.] *R. P. Boas, Jr.*

**Ciorănescu, Nicolas. Quelques formules de moyenne.** Bull. Math. Phys. Éc. Polytech. Bucarest **10** (1938–39), 27–31 (1940).

The author derives a number of mean-value theorems and inequalities by specializing the functions in

$$\iint\varphi(x_1,x_2)\frac{f(x_2)-f(x_1)}{x_2-x_1}dx_1dx_2=f'(\xi)\iint\varphi(x_1,x_2)dx_1\,dx_2,$$

where $\varphi\geqq 0$. *R. P. Boas, Jr.* (Providence, R. I.).

**Korevaar, J., van Aardenne-Ehrenfest, T., and de Bruijn, N. G. A note on slowly oscillating functions.** Nieuw Arch. Wiskunde (2) **23**, 77–86 (1949).

The function $L(x)$ is called slowly oscillating [Karamata, Mathematica, Cluj **4**, 38–53 (1930)] if (*) $L(\mu x)/L(x)\rightarrow 1$ as $x\rightarrow\infty$ for every positive $\mu$. For continuous slowly oscillating functions Karamata proved that (*) holds uniformly in any closed $\mu$-interval interior to $(0,\infty)$, via a representation for slowly oscillating functions. The authors give a simple direct proof of this uniformity property, first for continuous $L(x)$ and then for measurable $L(x)$. From it they derive Karamata's representation theorem. They show that some restriction on $L(x)$ is necessary for the uniformity of (*), by constructing an $L(x)$ for which (*) holds for each $\mu$ but does not hold uniformly in any interval. They put $\log L(e^x)=f(x)$, $\log\mu=\lambda$, and work with the class of $f(x)$ for which $f(x+\lambda)-f(x)\rightarrow 0$, $x\rightarrow\infty$; Karamata's representation takes the simplified form $f(x)=c(x)+\int_0^x\epsilon(t)dt$, where $\lim_{x\rightarrow\infty}c(x)$ exists and $\epsilon(x)=o(1)$. *R. P. Boas, Jr.*

**Korevaar, J. The zeros of the derivative of a function and its analytic character.** Math. Centrum Amsterdam. Rapport **ZW 1948-004**, 9 pp. (1948).

An expository lecture, with sketches of proofs and some improvement of details, and a comprehensive bibliography. For an earlier account of some of the problems involved, cf. Pólya, Bull. Amer. Math. Soc. **49**, 178–191 (1943); these Rev. **4**, 192. *R. P. Boas, Jr.* (Providence, R.I.).

## *Theory of Sets, Theory of Functions of Real Variables*

**Kuratowski, Casimir. Ensembles projectifs et ensembles singuliers.** Fund. Math. **35**, 131–140 (1948).

This paper is based on the author's hypothesis $H_p$ ("projective continuum hypothesis"): there exists a relation $x<y$ which orders the set of all real numbers of the interval [0, 1] in a transfinite sequence of the type $\Omega$ such that the plane set $E_{x,y}[x<y]$ is projective. (A set is called projective if it can be obtained from closed sets by finitely many projections and complements.) The author states that according to K. Gödel [cf. Proc. Nat. Acad. Sci. U. S. A. **24**, 556–557 (1938)] the hypothesis $H_p$ is consistent with the system of axioms of set theory. The author shows that applying the hypothesis $H_p$ the usual constructions for proving the existence of quite a few "singular" sets furnish projective sets. This is the case for Vitali's nonmeasurable set; for sets which with their complements are totally imperfect; for spaces with Lusin's property $\nu$; for Sierpiński's noncountable set $S$ $(\subset[0,1])$ which does not contain any nonmeasurable subset (this set $S$ is "always of first category" and has Sierpiński's property $\sigma$, that is, relative to $S$ every Borel subset is a $G_\delta$-set). *A. Rosenthal.*

**Sierpiński, Waclaw. Sur la division des types ordinaux.** Fund. Math. **35**, 1–12 (1948).

Among the results established here are the two following (which were stated without proof by A. Lindenbaum and A. Tarski [C. R. Soc. Sci. Lett. Varsovie. Cl. III. **19**, 299–330 (1926), p. 321]. Let $\alpha$ and $\beta$ be two order types. (1) If $\mu\cdot\alpha=\mu\cdot\beta$ for some ordinal number $\mu\neq 0$ then $\alpha=\beta$. (2) If $\alpha\cdot n=\beta\cdot n$ for some integer $n$ then $\alpha=\beta$. It is noted that (1) is no longer true when $\mu$ is an order type, e.g., if $\eta$ is the order type of the rational numbers in natural order, for $\eta\cdot 2=\eta\cdot 1$. It is noted that (2) remains true when $n$ is replaced by a nonlimiting transfinite ordinal but that it is false, e.g., when $n$ is replaced by $\omega$ since $1\cdot\omega=2\cdot\omega$. *J. Todd* (London).

**Sierpiński, Waclaw. Sur l'analycité de l'espace $D_\omega$ au sens de M. Menger.** Fund. Math. **35**, 208–212 (1948).

K. Menger [Jber. Deutsch. Math. Verein. **37**, 213–226 (1928)] called a separable metric space $M$ analytic if it could be represented in the form $M=\sum\prod A(n_1,n_2,\cdots,n_k)$ where the product is over $k=1,2,\cdots$, where the summation is over all infinite sequences $(n_1,n_2,\cdots)$ of integers, and where the sets $A$ are closed in $M$ and such that (*) each product $\prod A$ reduces to a single point (provided no set $A$ is empty). The author now defines, in the nonseparable set $D_\omega$ [M. Fréchet, Les Espaces Abstraits, Gauthier-Villars, Paris, 1928, p. 97], a system of closed sets $\{A(n_1,n_2,\cdots,n_k)\}$ which have the property (*) and for which $D_\omega=\sum\prod A$. *J. Todd* (London).

**Sierpiński, Waclaw. Sur un problème de la théorie générale des ensembles équivalent au problème de Souslin.** Fund. Math. **35**, 165–174 (1948).

The problem in question is whether a class $\mathbf{F}$ satisfying the three following conditions is necessarily countable. (1) If $X\epsilon\mathbf{F}$, $Y\epsilon\mathbf{F}$ then either $X\subset Y$, $X\supset Y$ or $XY=0$. (2) If $\mathbf{F}_1\subset\mathbf{F}$ and if $X\epsilon\mathbf{F}_1$, $Y\epsilon\mathbf{F}_1$ imply $XY=0$ then $\mathbf{F}_1$ is at most countable. (3) If $\mathbf{F}_2\subset\mathbf{F}$ and if $X\epsilon\mathbf{F}_2$, $Y\epsilon\mathbf{F}_2$ imply either $X\subset Y$ or $X\supset Y$ then $\mathbf{F}_2$ is at most countable and contains a maximal element. The author points out, in a footnote added in proof, that D. Maharam [Bull. Amer. Math. Soc. **54**, 587–590 (1948); these Rev. **9**, 573] has noted that the condition (3) can be replaced by (4) if $\mathbf{F}_2\subset\mathbf{F}$ and if $0\neq X\epsilon\mathbf{F}_2$, $0\neq Y\epsilon\mathbf{F}_2$ imply $XY\neq 0$ then $\mathbf{F}_2$ is at most countable. Conditions (2) and (4) are weaker than (2) and (3). *J. Todd* (London).

**Kurepa, Georges. L'hypothèse du continu et le problème de Souslin.** Acad. Serbe Sci. Publ. Inst. Math. 2, 26–36 (1948). (French. Serbian summary)

Partially ordered sets $E$ satisfying (1) $bE \leqq \aleph_0$ and the first or both of the two following conditions are studied: (2) every ordered subset of $E$ is well ordered, (3) for every $a\epsilon E$ the subset containing all predecessors of $a$ is ordered. Here $bE$ denotes the upper bound of the cardinals of subsets $E_1$ such that (1) $E_1$ is well ordered in the ordering of $E$, or (2) $E_1$ is well ordered in the opposite ordering, or (3) no two elements of $E_1$ are comparable. The results obtained [some of which were announced in C. R. Acad. Sci. Paris 205, 1196–1198 (1937)] indicate certain analogies between the two problems of the title. *J. Todd* (London).

**Bagemihl, F. A theorem on infinite products of transfinite cardinal numbers.** Quart. J. Math., Oxford Ser. 19, 200–203 (1948).

A. Tarski [Fund. Math. 7, 1–14 (1925)] raised the question as to the truth of the relation (*) $\prod_{\xi<\alpha}\aleph_{\sigma_\xi}=\prod\aleph_\lambda=\aleph_\lambda{}^a$, where $\alpha$ is a transfinite ordinal limiting number (and $a$ the corresponding cardinal) and $\{\sigma_\xi\}_{\xi<\alpha}$ is an increasing sequence of ordinals such that $\lim_{\xi<\alpha}\sigma_\xi=\lambda$. Tarski [and Hausdorff] established (*) in certain special cases, e.g., when $\sigma_\xi=\xi$. The author establishes (*) under certain rather complicated conditions which, roughly speaking, ensure that the numbers $\sigma_\xi$, for $\xi$ near $\alpha$, are not too far apart, e.g., if $\sigma_\xi=\mu+\xi$ for any ordinal $\mu$. *J. Todd* (London).

**Denjoy, Arnaud. L'ordre de nullité métrique des ensembles parfaits minces.** C. R. Acad. Sci. Paris 227, 928–931 (1948).

A linear perfect set $P$ is said to possess the property $A$ if $y-x$ describes a segment as $x, y$ describe $P$. The set $P$ possesses $H$ if $|\sigma|\geqq\min(|u|, |v|)$ for any two contiguous intervals $u, v$ and the interval $\sigma$ between them. Let $u_p$ $(0<p<n)$ be distinct contiguous intervals numbered from left to right, let $v_p$ be another such set, and $u_0$, $v_n$ semi-contiguous intervals to left and right of $P$. Suppose $u_p=v_p$ or $u_p$ to the left of $v_p$. Let $j_p$ be the least interval containing $u_p$ and $v_p$, let $\sigma_p$ be the interval between $u_{p-1}$ and $v_p$. The set $P$ is said to possess $B$ if $\max|\sigma_p|\geqq\min|j_p|$ for any such sets $u_p$, $v_p$. The author recalls his result $A\supset H$ [C. R. Congres Intern. Math. Strasbourg 1920, pp. 22–30 (1921); rediscovered by Mirimanoff, Fund. Math. 4, 118–121 (1923)] and proves $H\equiv B$. *H. D. Ursell* (Leeds).

**Borel, Émile. Sur la somme vectorielle de deux ensembles de mesure nulle dont un seul est parfait.** C. R. Acad. Sci. Paris 227, 790–792 (1948).

For a perfect set $F$ of measure zero in (0, 1), remove from (0, 1) the intervals contiguous to $F$ in decreasing order and let $b_n$ be the greatest residual interval after $n$ steps. The author defines the rarefaction $r=\limsup(-\log n/\log|b_n|)$. He considers also sets $G$ defined by open intervals $s_n$, $\sum|s_n|<\infty$, $x\epsilon G$ if $x\epsilon s_n$ for infinitely many $n$. For $G$ he defines $r'=\limsup(-\log n/\log|s_n|)$. He states (i) if $r+r'<1$ then the vector sum $G'$ of $F$ and $G$ is of measure zero, (ii) $G'$ is analogous to $G$ and of rarefaction $r''\leqq r+r'$. *H. D. Ursell* (Leeds).

**Neĭšuler, L. Ya. On tabularly-unique functions.** Doklady Akad. Nauk SSSR (N.S.) 61, 791–793 (1948). (Russian)

The author considers $f(x, y, z)$ of the type

$$f(x, y, z)=F[f_1(x, y), f_2(x, y)]. \tag{1}$$

If any other representation of this type:

$$f(x, y, z)=G[g_1(x, y), g_2(x, z)]$$

implies that the $f$'s are functions of the $g$'s, then $f(x, y, z)$ is called "tabularly-unique." All functions $f$ of the type (1) are tabularly unique except those capable of the representation

$$f(x, y, z)=H[h_1(x, y)+h_2(x, z)].$$

A similar result holds for functions of the type

$$f(x, y, z)=J_3(J_2(J_1(x, y), z), x).$$

*D. H. Lehmer* (Berkeley, Calif.).

**Neĭšuler, L. Ya. On the unique representation of functions of several variables by the superposition of functions of two variables.** Uspehi Matem. Nauk (N.S.) 3, no. 6(28), 205–210 (1948). (Russian)

The results of the paper reviewed above are extended to a function of $n$ variables represented by the superposition of $r$ functions of $k$ variables. Detailed results are obtained for $n=4$. *D. H. Lehmer* (Berkeley, Calif.)

**Climescu, Al. C. Notes d'analyse. II. L'interversion des dérivées partielles mixtes du second ordre.** Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 3, 526–531 (1948).

[For note I cf. same Bull. 2, 81–88 (1947); these Rev. 9, 503.] In the $(x, y)$-plane, denote by $C_n$ a smooth contour whose diameter approaches zero as $n\to\infty$, and which encloses an area $a_n$ containing the origin (0, 0). Let $f(x, y)$ be a continuous function such that $f_x$, $f_y$ exist in a neighborhood of (0, 0) and that $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ exist finite. The author proves that, in order that $f_{xy}(0, 0)=f_{yx}(0, 0)$, it is necessary and sufficient that

$$\lim_{n\to\infty} a_n^{-1}\int_{C_n} f_x(0, y)dx+f_y(x, 0)dy=0.$$

*L. C. Young* (Madison, Wis.).

**Zahorski, Zygmunt. Über die Menge der Punkte in welchen die Ableitung unendlich ist.** Tôhoku Math. J. 48, 321–330 (1941).

V. Jarník [same J. 37, 248–253 (1933)] showed that there exists a continuous function $f(x)$ whose derivative is infinite at every point of any given $G_\delta$-set $M$ of measure zero, while the derivates of $f(x)$ are finite in the complement of $M$. The author [answering a question proposed by Jarník, loc. cit.] shows that there exist everywhere differentiable functions $f(x)$ with the same property. *A. Rosenthal* (Lafayette, Ind.).

**Haupt, Otto, und Pauc, Christian. Über die Ableitung absolut additiver Mengenfunktionen.** Arch. Math. 1, 23–28 (1948).

The authors generalize results of the paper of R. de Possel [J. Math. Pures Appl. (9) 15, 391–409 (1936)] on the differentiation of totally additive set functions $F$ defined in a $\sigma$-field $\mathfrak{K}$. In order that the derivative of $F$ exist almost everywhere (with respect to a given measure $m$), de Possel assumed $F$ to be $m$-continuous. The authors succeed in deleting the assumption of $m$-continuity of $F$ by adding a further condition ("Axiom (U)" = "Umfassungsaxiom"), concerning the sets $V$ employed for the differentiation, to the system of assumptions used by de Possel. Let $\mathfrak{B}$ be the system of the sets $V$. With almost every point $a$, subsystems $\mathfrak{B}_a$ of $\mathfrak{B}$ are associated. Axiom (U) says: let $V$ be given;

then for almost all $x \epsilon V$ every $\mathfrak{B}_x$ is confinal with a subsystem of $\mathfrak{B}_x$ which is entirely contained in $V$. Besides axiom (U) another condition ("axiom (E)" = "Erzeugende-axiom") is added, in order to generalize, to non-$m$-continuous functions $F$, also the theorem that almost everywhere the derivative equals an integrand, i.e., in order to obtain the theorem: the derivative of $F$ almost everywhere equals a point function $f$ for which $\int_M f dm$ is the $m$-continuous part of $F(M)$. A result of this type had already been obtained by de Possel [C. R. Acad. Sci. Paris **224**, 1137–1139, 1197–1198 (1947); these Rev. **8**, 572] by means of another (less simple) additional condition. According to an example of de Possel, axiom (U) alone would not suffice for this purpose. Axiom (E) says: in the smallest $\sigma$-system over $\mathfrak{B}$, a $\sigma$, $d$-system $\mathfrak{D}$ (i.e., the sums of countably many sets of $\mathfrak{D}$ and the intersections of finitely many sets of $\mathfrak{D}$ have to belong to $\mathfrak{D}$) is contained, which is a "(Borel) generator" ("Erzeugende") for $\mathfrak{K}$, i.e., $\mathfrak{K}$ has to be contained in the smallest $\sigma\delta$-system over $\mathfrak{D}$. *A. Rosenthal* (Lafayette, Ind.).

**Salem, R. Sur les sommes Riemanniennes des fonctions sommables.** Mat. Tidsskr. B. **1948**, 60–62 (1948).

Let $f(x)$ be an $L$-integrable function of period 1, and let $M_n(f, x) = n^{-1}\sum_{k=1}^{n} f(x+k/n)$. It was shown by Jessen [Ann. of Math. (2) **35**, 248–251 (1934)] that, if $n_{k+1}$ is always divisible by $n_k$, then $M_{n_k}(f, x)$ tends to $\int_0^1 f(t)dt$ for almost every $x$. It is now shown that the conclusion holds for any sequence $\{n_k\}$ such that (*) $\sum(\log n_k)^{-1-\delta} < \infty$ for some $\delta > 0$, provided $f$ satisfies the condition

$$\int_0^1 |f(x+h) - f(x)| dx = O(\log 1/h)^{-1-\epsilon}$$

for some $\epsilon > \delta$. It is to be observed that (*) holds for any lacunary sequence $n_k$, that is, if $n_{k+1}/n_k > q > 1$ for all $k$. *A. Zygmund* (Chicago, Ill.).

**Hadwiger, H. Une mesurabilité moyenne pour les ensembles de points.** Fund. Math. **34**, 293–305 (1947).

Elementary properties of von Neumann means [Trans. Amer. Math. Soc. **36**, 445–492 (1934)] of the characteristic functions of subsets of Euclidean $n$-space and of the $n$-dimensional torus. *H. Federer* (Providence, R. I.).

**Tolstov, G. P. On the interchange of integrations.** Doklady Akad. Nauk SSSR (N.S.) **63**, 3–6 (1948). (Russian)

A function $f$ defined for $(x, y)$ in a rectangle $R$ is said to satisfy condition $(F)$ if over every rectangle $r \subset R$ the iterated integrals of $f$ exist and are equal, the integrals with respect to $x$ and to $y$ being Denjoy integrals. If $F$ exists for which $\partial^2 F/\partial x\partial y = \partial^2 F/\partial y \partial x = f$ for all $(x, y)$ in $R$, then $f$ satisfies condition $(F)$. When the product of $f$ and the characteristic function of a subset $E$ of $R$ satisfies condition $(F)$, the common value of the iterated integrals of the product defines an integral $I(f; E)$. Theorems are stated (without proof) connecting the existence of this integral over sets $G$ bounded by sectionally monotone curves $C$ with the line integrals around $C$ of $Pdx + Qdy$, where $P(x, y) = \int_c^y f(x, y)dy$, $Q(x, y) = \int_a^x f(x, y)dx$. As application, it is stated that, if $u$, $v$ are bounded and have all first-order partials finite everywhere in $R$, and $\partial u/\partial y = \partial v/\partial x$ almost everywhere, then there exists $F$ such that $\partial F/\partial x = u$, $\partial F/\partial y = v$; a generalization of Morera's theorem is also stated. *E. J. McShane* (Charlottesville, Va.).

**Stone, M. H. Notes on integration. IV.** Proc. Nat. Acad. Sci. U. S. A. **35**, 50–58 (1949).

[For note III cf. same Proc. **34**, 483–490 (1948); these Rev. **10**, 239.] In this note the author concludes, for the present, the outline of his theory of integration. He adjoins to his postulates an assumption which in the more usual measure-theoretic language is tantamount to the requirement of $\sigma$-finiteness and he proves the analogues of the Lebesgue decomposition theorem and the Radon-Nikodym theorem. The proof of the latter is a modification of von Neumann's proof (using linear functionals in the appropriate Hilbert space).

The last part of the note is devoted to a comparison of the author's theory with the [as yet unpublished] theory of integration due to Bourbaki. The principal difference between the two theories is that the latter assumes an inequality (subadditivity) for directed systems of functions which the former postulates for sequences only. It turns out that (in all cases to which the Bourbaki theory applies) the Bourbaki integral is an extension of the Stone integral; the difference between the two is similar to the difference between Borel and Baire measures. Using this fact it is possible within the framework of general integration theory to prove completely the theorem on the representation of positive linear functionals on the space of all those continuous functions on a locally compact space which vanish in a neighborhood of infinity, i.e., to go from the Baire measure that the author obtained earlier to a regular Borel measure. *P. R. Halmos* (Chicago, Ill.).

**McShane, E. J. Remark concerning integration.** Proc. Nat. Acad. Sci. U. S. A. **35**, 46–49 (1949).

The results of this paper are contained in Stone's fourth note on integration [cf. the preceding review]. The author's modification of Stone's theory coincides with the Bourbaki theory; the author's purpose in introducing it was to extend Stone's version of the representation theorem for positive linear functionals so as to obtain a Borel measure, instead of a Baire measure, exactly as described above. *P. R. Halmos* (Chicago, Ill.).

**Halmos, Paul R. On a theorem of Dieudonné.** Proc. Nat. Acad. Sci. U. S. A. **35**, 38–42 (1949).

Let $\mathbf{A}$ be an abstract Boolean $\sigma$-algebra and let $\mu$ be a finite positive measure on $\mathbf{A}$. Then $\mu$ can be considered to be defined on the Boolean algebra $\mathbf{E}$ of open-closed sets of the Stone space $Y$ associated with $\mathbf{A}$, and $\mu$ can be extended to the $\sigma$-algebra $\mathbf{B}$ generated by $\mathbf{E}$. The author calls such a pair $(Y, \mu)$ a Kakutani space and the sets of $\mathbf{B}$ are called Baire sets. Every set $B \epsilon \mathbf{B}$ determines a unique $E \epsilon \mathbf{E}$ such that $(B \Delta E) = 0$, where $B \Delta E$ is the symmetric difference $(E-B) \cup (B-E)$. A subalgebra $\mathbf{J}$ which contains with any $B$ the associated open-closed set $E$ is called full. The author gives a new proof of a theorem of Dieudonné, which in terms of the above notation can be stated as follows. If $(Y, \mu)$ is a Kakutani space and if $\mathbf{J}$ is a full Boolean $\sigma$-subalgebra of $\mathbf{B}$, then there exist a Kakutani space $(Z, \nu)$, a continuous mapping $\pi$ from $Y$ onto $Z$, and, for each $z$ in $Z$, a measure $\mu^z$ on the relative Baire sets of $\pi^{-1}(z)$ such that the set transformation induced by $\pi$ is a one-to-one mapping from $\mathbf{J} \cap \mathbf{E}$ onto the class $\mathbf{F}$ of all open-closed sets in $Z$ and such that, for every open-closed set $E$ in $Y$, $\mu(E) = \int \mu^z(E \cap \pi^{-1}(z))d\nu(z)$.

The author wishes to call attention to the following errata in the paper as published. P. 39, line 13 from the bottom:

Insert "non empty" before "$E$ in $\mathbf{E}$" and "$U$ in $\mathbf{U}$," and add the sentence: "A Baire measure $\mu$ in $Y$ is dense if $\mu(K)=0$ whenever $K$ is a compact Baire set with an empty interior (and hence $\mu(B)=0$ whenever $B$ is a Baire set with an empty interior)"; p. 39, line 11 from the bottom and line 9 from the bottom: Insert "dense" between "positive" and "Baire"; p. 41: Replace lines 5–12 from the bottom by: "If $\mu^z$ is defined, for each Baire set in $\pi^{-1}(z)$ by $\mu^z(B\cap\pi^{-1}(z))=q(B,z)$, then the desired integral formula follows from the defining properties of the functions $p$ and $q$. It remains only to prove the uniqueness of this definition, i.e., to show that the complement of $\pi^{-1}(z)$ has inner measure 0 with respect to $q^z$." *L. H. Loomis* (Cambridge, Mass.).

**Nikodym, Otton Martin. Tribus de Boole et fonctions mesurables. Tribu spectrale d'une fonction.** C. R. Acad. Sci. Paris **228**, 37–38 (1949).

**Nikodým, Otton Martin. Tribus de Boole et fonctions mesurables. Transformations équimesurables.** C. R. Acad. Sci. Paris **228**, 150–151 (1949).

The author announces [without proofs] a decomposition theorem for measure algebras and its applications to what may be called the spectral theory of measurable functions. He indicates the analogue in this theory of the concept of multiplicity for (not necessarily discrete) spectra.

*P. R. Halmos* (Chicago, Ill.).

**Schaerf, H. M. On the continuity of measurable functions in neighborhood spaces. II.** Portugaliae Math. **7**, 91–92 (1948).

As a supplement to an earlier paper [Portugaliae Math. **6**, 33–44 (1947); these Rev. **9**, 18] it is noted that Lusin's theorem holds for any totally $\sigma$-finite regular measure in a neighborhood space. *J. C. Oxtoby* (Bryn Mawr, Pa.).

**Dubrovskiĭ, V. M. On properties of absolute continuity and equi-continuity.** Doklady Akad. Nauk SSSR (N.S.) **63**, 483–486 (1948). (Russian)

The author announces extensions of results described in previous communications [Rec. Math. [Mat. Sbornik] N.S. **20**(62), 317–329 (1947); same Doklady (N.S.) **58**, 737–740 (1947); these Rev. **9**, 19, 275; for terminology, see the latter review]. Let $A$ be a fixed abstract set and let $\mathfrak{M}$ be a family of subsets of $A$, containing $A$ and closed under the formation of complements and countable unions. Let $\{\Phi_\alpha\}$, $\alpha\epsilon A$, be a family of finite, real-valued, countably additive set-functions each defined on $\mathfrak{M}$. The family $\{\Phi_\alpha\}$ is said to be uniformly equicontinuous with respect to a family $\{M_\beta\}$, $\beta\epsilon B$, of bases of $\{\Phi_\alpha\}$ if, for every $\epsilon>0$, there exists a $\delta>0$ such that $M_\beta(X)<\delta$ implies $|\Phi_\alpha(X)|<\epsilon$ for all $X\epsilon\mathfrak{M}$, $\alpha\epsilon A$, and $\beta\epsilon B$. A single function $\Phi_\alpha$ is said to be uniformly absolutely continuous with respect to the family $\{M_\beta\}$ if for every $\epsilon>0$ there exists a $\delta>0$ such that $M_\beta(X)<\delta$ implies $|\Phi_\alpha(X)|<\epsilon$ for every $\alpha\epsilon A$ and $X\epsilon\mathfrak{M}$. A basis $M$ for the family $\{\Phi_\alpha\}$ is said to be regular if $\overline{\Phi}_\alpha(X)=0$ for all $\alpha\epsilon A$ implies that $M(X)=0$. Let $g(\beta,x)$ be a real-valued function defined on $A$, depending upon the index $\beta$ ($\beta\epsilon B$), and let $\{M_\beta\}$ be a family of nonnegative, countably additive, finite, real-valued set-functions defined on $\mathfrak{M}$. Suppose that the integrals $\int_A g(\beta,x)dM_\beta$ all exist; these integrals are said to converge uniformly with respect to the index $\beta$ if, for every $\epsilon>0$, there exists a real number $N(\epsilon)>0$ such that $\int_E(g(\beta,x)-N)dM_\beta<\epsilon$, where $E$ is the set of $x$ such that $x\epsilon A$, $g(\beta,x)>N$, and the number $N$ is independent of $\beta$.

The following theorem is proved. (I) Let $\{\Phi_\alpha\}$ be a family of set-functions as described above. If the functions $\Phi_\alpha$ are all equicontinuous with respect to one basis $M$, then they are all equicontinuous with respect to any other basis $M^*$. The author states that the proof presented here is briefer although less perspicuous than a previously published proof of the same theorem.

The following three theorems are stated without proof. (II) Let $\{M_\beta\}$ be a family of nonnegative countably additive set-functions defined on $\mathfrak{M}$. Suppose that $M_{\beta_1}(X)=0$ if and only if $M_{\beta_2}(X)=0$, for all $\beta_1$, $\beta_2\epsilon B$ and $X\epsilon\mathfrak{M}$. Then, if one function $M_{\beta_0}$ is absolutely continuous with respect to all of the others, every function $M_\beta$ is absolutely continuous with respect to all of the others. (III) Let $\{M_\beta\}$ be a family of set-functions satisfying the requirements of (II). If $\beta$ and $\beta_1$ are any two elements of $B$, then, by the Radon-Nikodym theorem, there is a real-valued function $f(\beta_1,\beta,x)$ on the set $A$, summable with respect to $M_\beta$, such that $M_{\beta_1}(X)=\int_X f(\beta_1,\beta,x)dM_\beta$ for all $X\epsilon\mathfrak{M}$. Then, if this integral converges uniformly with respect to $\beta$ for some fixed $\beta_1$, it converges uniformly with respect to $\beta$ for all values of $\beta_1$. (IV) Let $\{\Phi_\alpha\}$ be a family of countably additive, finite, real-valued set functions on $\mathfrak{M}$, and let $\{M_\beta\}$ be a family of regular bases of $\{\Phi_\alpha\}$ such that the functions $\Phi_\alpha$ are equicontinuous with respect to each $M_\beta$. In order that the functions $\Phi_\alpha$ be uniformly equicontinuous with respect to all of the functions $M_\beta$, it is necessary and sufficient that each $M_{\beta_0}$ be uniformly absolutely continuous with respect to all of the other functions $M_\beta$. *E. Hewitt.*

**Federer, Herbert. Essential multiplicity and Lebesgue area.** Proc. Nat. Acad. Sci. U. S. A. **34**, 611–616 (1948).

The purpose of the paper is to announce results which represent generalizations to $n$ dimensions of results established in previous literature only in the two-dimensional case. The topological index, which has been used as a principal tool in the 2-dimensional case, is replaced by the degree of the mappings involved, and more precisely the degree relative to the mapping operating from the frontier is expressed in terms of the induced homomorphisms for the cohomology groups, in the appropriate dimension, relative to the regions themselves modulo the frontier. In brief, the technique of algebraic topology is fully applied. The expert in the theory of the Lebesgue area should note that the author deals essentially with the "flat case."

*T. Radó* (Columbus, Ohio).

**Eggleston, H. G. Intersections of sets in Euclidean space.** J. London Math. Soc. **23**, 92–100 (1948).

The prototype of the generalizations presented in this paper is the following elementary fact. In the Euclidean plane, for example, let $S$ be a closed and bounded set and $P$ a fixed point. Let $L(\phi,S)$ denote the linear measure of the intersection of $S$ with the straight line through $P$ which makes the angle $\phi$ with a fixed direction. Then it is well known that $L(\phi,S)$ is lower semicontinuous as a function of $\phi$. The author considers generalizations of the following character. Let $A(t)$ be a set which depends upon the real variable $t$, and which is a continuous function of $t$ in a certain plausible sense. Let $L(t,S)$ denote the Carathéodory measure of the intersection of $A(t)$ with a fixed open set in Euclidean $n$-space, and let measure be taken in the appropriate dimension. Then, under various types of suitable restrictions, $L(t,S)$ is a lower semicontinuous function of $t$. Indeed, if $A(t)$ is suitably restricted, then by suitably choosing $S$ the function $L(t,S)$ can be made to coincide with any assigned nonnegative, bounded lower semicontinu-

ous function $G(t)$. Further generalizations are concerned with the intersection of two continuous families of sets. Among the applications, one of the corollaries yield a characterization of Jordan curves. *T. Radó.*

**Cesari, Lamberto. Parametrizzazione delle superficie continue di area finita secondo Lebesgue.** Ann. Mat. Pura Appl. (4) **26**, 301–374 (1947).

Let $A(S)$ be the Lebesgue area of the surface $S: x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $(u, v)\epsilon R$, where $R$ is a bounded, simply connected Jordan region, and the coordinate functions are continuous in $R$. Assume that $A(S)<\infty$. The author establishes the result that $S$ admits then of a representation, of the form indicated above, such that $A(S)$ is given by the classical area-integral in terms of ordinary Jacobians. Indeed, this representation may be chosen as a quasi-conformal representation. The expert in the field should note carefully that quasi-conformal representations are more general than the generalized conformal representations in the sense of Morrey [for details, see the reviewer's book, Length and Area, Amer. Math. Soc. Colloquium Publ., v. 30, New York, 1948; these Rev. **9**, 505].

*T. Radó* (Columbus, Ohio).

**Ward, A. J. The differentiable parametrization of a surface.** Proc. London Math. Soc. (2) **50**, 409–422 (1949).

Consider a point $O$ on a two-dimensional manifold in three-dimensional Euclidean space having a closed neighborhood $S$ which is a topological 2-cell. Condition $A$ is said to be satisfied if there exists a function $\psi(r)$ defined for positive values of $r$, converging to zero with $r$, and having the following property. Let $P$ be any point on $S$ and $C$ any simple closed curve on $S$, such that $P$ and all the points of $C$ lie within $r$ of $O$. On any plane project $P$ and $C$ orthogonally into a point $\bar{P}$ and a continuous closed curve $\bar{C}$. Then $\bar{C}$ may be deformed in that plane by moving each point at most $\psi(r)$ times the distance of the corresponding point of $C$ from $O$ into a curve which does or does not surround $\bar{P}$ according as $C$ does or does not surround $P$ on $S$. In case $C$ passes through $O$, the plane onto which $P$ and $C$ are projected must not be perpendicular to $OP$. It is first shown that condition $A$ guarantees the existence of a tangent plane to $S$ at $O$. Then the existence of a parametrization for $S$ which is differentiable at $O$ is established.

*P. V. Reichelderfer* (Columbus, Ohio).

## *Theory of Functions of Complex Variables*

**Pastidès, N. Sur les séries entières à rayon de convergence nul.** Bull. Sci. Math. (2) **72**, 107–115 (1948).

The author studies formal power series over the complex field, with constant term zero. A product $ST$ is defined to be the series obtained from $S(x)$ by replacing $x$ by $T(x)$, expanding and then collecting terms. The product is associative, there is an identity $I$, and for any $S$ there is a unique inverse $S^{-1}$ such that $SS^{-1}=S^{-1}S=I$. Let $r(S)$ be the radius of convergence of $S$. If $r(S)$ and $r(T)$ are both positive, then $r(ST)$ is positive; if one is positive and the other zero, then $r(ST)=0$; however, if both are zero, then $r(ST)$ may be zero, positive or infinite. If $r(S)$ is positive (zero) then $r(S^{-1})$ is positive (zero). Some remarks are made about power series with nonzero constant term.

*R. C. Buck* (Providence, R. I.).

**Bradley, F. W. Analytic solutions of certain functional equations.** Proc. Math. Phys. Soc. Egypt **3** (1947), 37–45 (1948).

Etude des solutions analytiques des équations fonctionnelles (1) $\psi(f(z))=a\psi(z)$, (2) $g(f(z))=f(g(z))$, où $f(z)$ est une fonction donnée et $\psi$ et $g$ respectivement la fonction inconnue. Les hypothèses sont: $f(z)$ holomorphe et univalente dans un domaine $D$ contenant le segment $0\leqq x<1$, $x=\Re(z)$; $f(x)$ réelle croissante et continue sur le segment $0\leqq x\leqq 1$; $f(0)=0$, $f(1)=1$, $f'(0)\neq 1$; $f(z)\neq z$ en tout point de $D$ autre que $z=0$ et $z=1$. Par la méthode des fonctions majorantes, l'auteur montre que: (I) $f(z)$ peut se mettre sous la forme $\psi^{-1}(a\psi(z))$, $a=f'(0)$, $\psi$ étant unique sous la condition $\psi'(0)=1$, holomorphe dans un domaine $\Delta$ contenant le segment $0\leqq x\leqq 1-\delta$, si petit que soit $\delta>0$ et $\psi(0)=0$; en outre $\psi(x)$ est réelle et croissante pour $0\leqq x<1$, et tend vers l'infini lorsque $x$ tend vers 1. La fonction $\psi$ est donc solution de (1) et il s'ensuit que: (II) L'équation (2) a une solution analytique unique pour laquelle $g'(0)$ a une valeur donnée $\alpha$, on a $g=\psi^{-1}(\alpha\psi(z))$.

L'auteur considère aussi le groupe des fonctions $f_\lambda=\psi^{-1}(a^\lambda\psi(z))$. Il signale que certains résultats de Koenigs [Ann. Sci. École Norm. Sup. (3) **1**, supplément, 1–42 (1884)] sont équivalents à son énoncé (I). La fonction $\psi$ est d'ailleurs la fonction appelée fonction de Koenigs par les auteurs qui ont traité de l'itération [Julia, Fatou, etc.], fonction qui permet de définir l'itération analytique, c'est-à-dire $f_\lambda$.

*G. Valiron* (Paris).

**Dufresnoy, Jacques. Autour du théorème de Phragmén-Lindelöf.** Bull. Sci. Math. (2) **72**, 17–22 (1948).

The author establishes several theorems of the Phragmén-Lindelöf type of which the following is characteristic. Let $U$ be subharmonic in $\Im(z)>0$ and satisfy the Phragmén-Lindelöf boundary condition. Let $\lambda=\lim_{r\to\infty}(\pi r)^{-1}\int_0^\pi U(r, \theta)\sin\theta d\theta$ and $\alpha=\liminf_{r\to\infty}(\pi r^2)^{-1}\int_0^\pi U^+(r, \theta)\sin\theta d\theta$. If $\lambda, \alpha<+\infty$, then (1) $\limsup_{r\to\infty}(\pi r^2)^{-1}\int_0^\pi|U(r, \theta)|\sin\theta d\theta<+\infty$; (2) $\lim_{r\to\infty}(\pi r^2)^{-1}\int_0^\pi U(r, \theta)\sin 2\theta d\theta$ exists and is finite (denoted hereafter by $\mu$); (3) $|\mu|\leqq 4\alpha$; (4) $U(r,\theta)\leqq 2\lambda r\sin\theta+2\mu r^2\sin 2\theta$. The proofs are based upon the Poisson integral formula for a semicircle (for harmonic functions vanishing continuously on the diameter). The author has overlooked the work of A. Dinghas [S.-B. Preuss. Akad. Wiss. **1938**, 32–48], which is related in method but of more limited scope, and that of Ahlfors [Trans. Amer. Math. Soc. **41**, 1–8 (1937)].

*M. Heins* (Providence, R. I.).

**Kobori, Akira. Zur Theorie der mehrwertigen Funktionen.** Jap. J. Math. **19**, 301–319 (1947).

Let $\zeta(z)=(c_0+c_1z+\cdots)/z^p$ be regular and $p$-valent in $0<|z|<1$, $c_0\neq 0$, and let $\{z^p\zeta(z)\}^{\lambda/2p}=b_0+b_1z+\cdots$ be regular in $|z|<1$, $\lambda>0$; then $\sum_{\nu=1}^\infty(2\nu-\lambda)|b_\nu|^2\leqq\lambda|b_0|^2$. The author uses this result to prove that if

$$w(z)=z^p+a_{p+1}z^{p+1}+a_{p+2}z^{p+2}+\cdots$$

is regular and $p$-valent in $|z|<1$, then (A) $|a_{p+1}|\leqq 2p$, (B) $|(p+1)a_{p+1}^2-2pa_{p+2}|\leqq 2p^2$, (C) $(1.0604\cdots)^p\,|w(z)|\geqq|z|^p/(1+|z|)^{2p}$, (D) $w(z)$ is starlike for $|z|<1/\sqrt{3}$, (E) $\{w(z)\}^{1/p}$ is univalent in $|z|<1/\sqrt{2}$. The result (A) was obtained earlier by T. Yosida [Proc. Imp. Acad. Tokyo **20**, 16–19 (1944); these Rev. **7**, 288]. A misprint occurs in the proof of lemma 2, where in the expression for $\Phi'''(x)$ an exponent 2 has been omitted from the next to last bracketed term.

*A. W. Goodman* (Lexington, Ky).

**Mathéev, A. Sur les fonctions holomorphes dans le cercle-unité, dont les zéros ont tous leurs points limites sur la frontière.** C. R. Acad. Bulgare Sci. Math. Nat. 1, no. 1, 29–32 (1948).

A false proof that a class of functions

$$R(z)=\prod_{i=1}^{n} z_i(z_i-z)/(1-\bar{z}_i z), \qquad |z_i|<1,$$

is normal in $|z|<1$. The result is trivial, since $|R(z)|<1$, $|z|<1$. *W. K. Hayman* (Exeter).

**Bloch, André. Sur les fonctions bornées à zéros multiples, les fonctions à valeurs ramifiées, et les couples de fonctions soumises à certaines conditions.** Bull. Sci. Math. (2) 72, 72–75 (1948).

The author announces three theorems of which the following is the simplest. Suppose that there exists $f(z)$, such that $|f(z)|<1$, $|z|<1$, $f(a_i)=l_i$, $i=1, \cdots, p$, where the $a_i$, $l_i$ are assigned and $f(z)$ has all its zeros of order at least $m$. Then there exists $f_0(z)$, satisfying the above conditions and having its zeros of order exactly $m$. Methods of proof are indicated. *W. K. Hayman* (Exeter).

**Breusch, Robert. On the distribution of the values of $|f(z)|$ in the unit circle.** Bull. Amer. Math. Soc. 54, 1109–1114 (1948).

Soit $f(z)=1+a_1z+\cdots$ holomorphe pour $|z|\leqq 1$, $f(z)\not\equiv 1$, soit $A(f)$ l'aire de l'ensemble des points du cercle unité où $|f(z)|\geqq 1$, et $\alpha$ et $\beta$ les plus grand nombres positifs ou nuls tels que, pour toute $f(z)$, on ait $\alpha\leqq A(f)\leqq\pi-\beta$. L'auteur montre que $\alpha=\beta=0$; cette propriété restant vraie pour l'ensemble restreint des polynomes $\prod_1^n(z-z_\nu)$ avec $\prod|z_\nu|=1$. Si $N$ est réel supérieur à 3 et si $|z|\leqq 1$, on a $|1+Nz+z^2|\geqq|Nz|-2>1$ si $|Nz|>3$, donc

$$A(1+Nz+z^2)>\pi-N^{-2}\pi$$

et $\beta=0$ pour ces polynomes. Pour montrer que $\alpha=0$, l'auteur introduit d'abord une exponentielle dont il prend ensuite une valeur approchée polynomiale et s'appuie sur ce lemme: soit $F(z)=b_1z+\cdots\not\equiv 0$, holomorphe pour $|z|\leqq 1$ et $B$ l'aire de l'ensemble $B'$ des points $|z|\leqq 1$ en lesquels la partie réelle de $F(z)$ est positive; si $n$ est positif et assez grand, et si $u(z)$ est le coefficient de $i$ dans $F(z)$, on a $\cos(nu(z))<0$ dans un sous ensemble de $B'$ dont l'aire est au moins $B/3$. D'autres résultats relatifs aux polynomes pour lesquels, en outre des hypothèses ci-dessus, les $|z_\nu|$ sont voisins de 1, sont donnés sans démonstrations. *G. Valiron* (Paris).

**Collingwood, E. F. Inégalités relatives à la distribution des valeurs d'une fonction méromorphe dans le cercle unité.** C. R. Acad. Sci. Paris 227, 813–815 (1948).

L'auteur applique la méthode des notes précédentes [mêmes C. R. 227, 615–617, 709–711, 749–751 (1948); voir ces Rev. 10, 244] au cas hyperbolique, $f(z)$ méromorphe pour $|z|<1$. Les notations étant les mêmes, voici le théorème principal. Si $T(r, f)$ n'est pas bornée et si $\sigma(r)$ est non croissante, on a trois cas possibles: (1) $r=1$ n'est pas point limite de $CE(a, \sigma(r))$, alors $\Delta(a)\leqq\bar{I}(\sigma(r), f)+\bar{P}$, $\delta(a)\leqq\underline{I}(\sigma(r), f)+\bar{P}$, $\delta(a)\leqq\bar{I}(\sigma(r), f)+\underline{P}$, $\bar{P}$ et $\underline{P}$ désignant respectivement les limites supérieure et inférieure pour $r\rightarrow 1$ de $\pi P(r, a, \sigma(r))/T(r, f)$; (2) $r=1$ est point limite de $E(a, \sigma(r))$ et de $CE(a, \sigma(r))$, alors, si $r'$ appartient à $E(a, \sigma(r))$, on a

$$\delta(a)\leqq\pi \liminf_{r'\rightarrow 1}\frac{P(r', a, \sigma(r))}{T(r', f)}+\bar{I}\leqq\bar{P}+\bar{I};$$

(3) $r=1$ n'est pas point limite de $E(a, \sigma(r))$. Il s'ensuit notamment que, si $\delta(a)>0$ et $I(\sigma(r), f)<\delta(a)$, toute circonférence $|z|=r$ intercepte lorsque $r\rightarrow 1$, soit un domaine $G(r, a, \sigma(r))$ non borné, soit un $G(r, a, \sigma(r))$ borné où la valence est de l'ordre de $T(r, f)$. *G. Valiron* (Paris).

**Combes, Jean. Sur le théorème de Landau-Carathéodory.** C. R. Acad. Sci. Paris 228, 41–42 (1949).

Remarks on the fact that, if $f(z)=a_0+a_1z+\cdots$, $a_1\neq 0$, is regular in $|z|<R$ and takes none of a set $E$ of complex values, we have $R|a_1||\omega'(a_0)|\leqq 1$, where $\omega(t)$ maps a complementary domain of $E$ onto $|z|<1$ so that $\omega(a_0)=0$. An extended result covering certain algebraic functions is also announced. *W. K. Hayman* (Exeter).

**Baganas, Nicolas. Sur les algébroïdes exceptionnelles ou quasi exceptionnelles pour une algébroïde donnée.** C.R. Acad. Sci. Paris 228, 533–534 (1949).

Announcement without proofs of theorems which extend to the theory of algebroid functions the theorems of Valiron concerning entire functions with exceptional or quasi-exceptional values [C. R. Acad. Sci. Paris 173, 1059–1061 (1921)]. *M. Heins* (Providence, R. I.).

**Pham, Tinh-Quat. Les fonctions entières périodiques.** Ann. Sci. École Norm. Sup. (3) 65, 11–70 (1948).

Let $f$ be an entire function of finite order and period $2\pi$. Let the zeros of $f$ in the fundamental strip $|x|\leqq\pi$ be $a_1, a_2, \cdots$. The zeros of $f$ are then the points $a_n+2k\pi$; if the convergence exponent of $\{a_n\}$ is $\rho$, then the genus of $f$ is $1+\rho$. If $f$ has only a finite number of fundamental zeros, then it may be represented as $\exp(F(z))\prod_1^N\sin(a_k-z)/2$. More generally, any function $f$ may be represented as the uniform limit of such functions, and any sequence $\{a_n\}$ is admissible. Any such function may also be expressed as $F(e^{-iz})G(e^{iz})$, where $F$ and $G$ are of order zero. Using these results, the author obtains rather sharp estimates of $\log|f(z)|$ on horizontal lines, and on circles, and discusses the growth of $f$, the distribution of its values, and the existence of asymptotic values and Borel directions. Example: no periodic entire function of finite order can have more than one asymptotic value. If $f$ is not a finite sum of exponentials, then every right angle contains a Borel direction. *R. C. Buck* (Providence, R. I.).

**Bernštein, S. N. The extension of properties of trigonometric polynomials to entire functions of finite degree.** Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 421–444 (1948). (Russian)

**Bernštein, S. N. A remark on my paper "The extension of properties of trigonometric polynomials to entire functions of finite degree."** Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 571–573 (1948). (Russian)

The author establishes a number of inequalities for entire functions $G_p(x)$ of finite degree $p$ (i.e., of exponential type $p$). The principal results are as follows. (1) If $|G_p(a_k)|\leqq 1$ for a sequence of real $a_k$ such that $0<a_{k+1}-a_k\leqq\pi/(p\mu)$, $\mu>1$, and $G_p(x)=O(|x|^m)$ for some $m$ as $|x|\rightarrow\infty$, then $G_p(x)$ is bounded for real $x$ and its least upper bound $g_\mu$ does not exceed $\sec\pi/(p\mu)$. (2) The order condition on $G_p(x)$ can be dropped if the $a_k$ form an arithmetic progression. (3) If $\mu^{-1}=1-N^{-1}$, where $N$ is an integer, then $g_\mu\leqq N^{-1}\sum_{k=1}^{n}\csc\frac{1}{2}(2k-1)\pi/N\sim 2\pi^{-1}\log N$, and this is essentially the best possible result. (4) The bound for $G_p(x)$ can be still further improved if $G_p(x)\geqq 0$ for real $x$.

(5) If $G_p(x)=o(|x|)$ as $|x|\to\infty$ and $|G_p(k\pi/p)|\leqq 1$, $k=0, \pm 1, \pm 2, \cdots$, then $|G_p(x+\frac{1}{2}\pi/p)+G_p(x-\frac{1}{2}\pi/p)|\leqq 8/\pi$ for $-\infty<x<\infty$, and this is a best possible result. A number of other results concern the best approximation to given functions by entire functions $G_p(x)$ and by various interpolation formulas. An appendix gives some lemmas on the degree of the product of two functions and on the degree of a function with a given density or upper density for its zeros. The second paper improves some of the results of this appendix.

The author has evidently overlooked some of the recent literature dealing with his problems. In particular, result (1), except for the estimate for $g_p$, is contained in a theorem of Duffin and Schaeffer [Amer. J. Math. 67, 141–154 (1945); these Rev. 6, 148], who assume $a_{k+1}-a_k\geqq\epsilon>0$; some condition of this sort seems to be essential; result (3), except for the constant $2/\pi$, was given by Boas and Schaeffer [Duke Math. J. 9, 879–883 (1942); these Rev. 4, 137] by a method essentially the same as the author's.

*R. P. Boas, Jr.* (Providence, R. I.).

**Leont'ev, A. F. On a class of functions defined by series of Dirichlet polynomials.** Uspehi Matem. Nauk (N.S.) 3, no. 4(26), 176–180 (1948). (Russian)

This is a summary, without proofs, of the results of the author's thesis. Let $\{\lambda_n\}$ denote a sequence of distinct complex numbers of nondecreasing modulus. The author quotes [without reference] the result that the condition $\limsup n/|\lambda_n|=\infty$ is both necessary and sufficient for the set $S$ of functions $\exp(-\lambda_n z)$ to be complete in every bounded region of the complex plane, i.e., for any holomorphic function to be represented by a uniformly convergent sequence of "Dirichlet polynomials" (1) $P_n(z)=\sum_{i=1}^{n}a_{ni}\exp(-\lambda_i z)$. If the set $S$ is not complete and a sequence of type (1) converges uniformly in some region $D$, either the region $D$ or the limit function must have special properties. The author states a number of theorems regarding these properties, with the hypotheses that the numbers $\lambda_n$ are positive, that the density $\sigma=\lim n/\lambda_n$ exists, that the region $D$ contains in its interior a closed vertical segment of length $2\pi\sigma$ on some line $z=x_0+iy$, and that (1) converges to $f(z)$ uniformly in $D$. Under these conditions (1) converges in some half-plane $x>\beta$ $(\beta<x_0)$. The quantities $\lim_{n\to\infty}a_{ni}=a_i$ exist; a second sequence of polynomials $P_n'$ converges to $f(z)$ if and only if $\lim a_{ni}'=\lim a_{ni}$ for each $i$, and in particular the Dirichlet series (2) $\sum a_i\exp(-\lambda_i z)$ converges to $f(z)$, if it converges. The sequence of polynomials $R_n(z)=\sum_{i=1}^{n}a_i\exp(-\lambda_i z)L_n(\lambda_i)$, where $L_n(z)=\prod_{j=n+1}^{\infty}[1-(z/\lambda_j)^2]$ converges to $f(z)$ in some half-plane $x>\alpha$ $(\alpha\leqq\beta)$. If the function $f(z)$ is not entire, every closed segment of length $2\pi\sigma$ on its axis of holomorphism contains a singularity of $f(z)$. There exists a sequence $\{m_k\}$ of positive integers, depending only on the sequence $\{\lambda_n\}$, such that the sequence of partial sums of order $m_k$ of the series (2) converges to $f(z)$ in the half-plane $x>\alpha$. If the quantity $\delta=\limsup \lambda_n^{-1}\log|1/L_0'(\lambda_n)|$ is finite, the series (2) converges in the half-plane $x>\alpha+\delta$. If $\delta=\infty$, there exists a function for which the series (2) diverges everywhere. A necessary and sufficient condition for (2) to converge in the half-plane $x>a$ is that the function $f(z)$ be bounded in every half-plane $x>a+\epsilon$ $(\epsilon>0)$. The author states that his results depend on the fact that, if the numbers $a_j$ are determined by the identity $L_0(x)=\sum a_jx^j$, the limit function of any uniformly convergent sequence of type (1) is a solution of the differential equation $\sum_{j=0}^{\infty}a_jf^{(j)}(z)=0$. The author states that his results and methods can be applied to obtain theorems on interpolation of entire functions of exponential type. For example, a necessary and sufficient condition for the existence of an entire function $\omega(z)$ of exponential type with $\omega(\pm\lambda_n)=a_{\pm n}$, $|a_{\pm n}|<\exp\{C|\lambda_n|\}$, is that $\limsup n/|\lambda_n|<\infty$ and $\limsup |\lambda_n|^{-1}\log|1/L_0'(\lambda_n)|<\infty$.

*R. P. Boas, Jr., and G. Piranian.*

**Džrbašyan, M. M. On the completeness of certain systems of analytic functions on linearly measurable sets.** Doklady Akad. Nauk SSSR (N.S.) 62, 581–584 (1948). (Russian)

Let $E$ be a linear, linearly measurable set in the complex plane. The author considers the closure of sets $\{z^{\lambda_n}\}$ in $L_2(E)$. Write $m(r)$ for the measure of the part of $E$ outside $|z|<r$. Let $h(r)$ be a function of the form $c+\int_a^r t^{-1}g(t)dt$ $(c>0, 0\leqq g(t)\uparrow\infty)$. The following results are typical [no proofs are given]. (1) Let $E$ lie on $n$ half-lines dividing the plane into connected regions $G_k$ such that each $G_k$ contains an angle of opening $\pi/\alpha_k$ $(\frac{1}{2}<\alpha_k<\infty)$. Let $\alpha=\max\alpha_k$. If $-\log m(r)\geqq h(r)$ and $\int h(r)r^{-1-\alpha}dr=\infty$, then $\{z^n\}$ $(n=0, 1, 2, \cdots)$ is complete in $L_2(E)$. (2) Suppose $\lambda_1>0$, $\lambda_{n+1}-\lambda_n\geqq c>0$. Put $\psi(r)=\exp\{2\sum_{\lambda_n<r}\lambda_n^{-1}\}$. Let $E$ lie on a finite number of nonintersecting half-lines all in $-\pi<\gamma\leqq\arg z\leqq\delta<\pi$. If $\psi(r)r^{-(\delta-\gamma)/\pi}\geqq k(r)$, $k(r)\uparrow$, $-\log m(r)\geqq\mu r^p$ $(\mu, p>0)$, and $\int k^p(r)r^{-2}dr=\infty$, then $\{z^{\lambda_n}\}$ $(|\arg z|<\pi)$ is complete in $L_2(E)$. (3) Let $E$ lie on the boundary of the region $x\geqq x_0$, $bx^\alpha\leqq y\leqq ax^\alpha$ $(b<a, \alpha<1)$. Let $f(z)$ be an integral function of order $\rho$, type $\sigma$, $\{a_n\}$ a sequence of complex numbers, $n(t)=\sum_{|a_n|<t}1$. If

$$-\log m(r)\geqq\mu\exp\{\pi r^{1-\alpha}(a-b)^{-1}(1-\alpha)^{-1}\}$$

$(\mu>0)$ and

$$\limsup_{r\to\infty} r^{-\rho}(\log r)^{-\rho/(1-\alpha)}\int_0^r t^{-1}n(t)dt >\sigma(\rho(a-b)(1-\alpha)/\pi)^{\rho/(1-\alpha)},$$

then $\{g(a_nz)\}$ is complete in $L_2(E)$.

*W. H. J. Fuchs.*

**Volkovyskiĭ, L. I. The determination of the type of certain classes of simply connected Riemann surfaces.** Mat. Sbornik N.S. 23(65), 229–258 (1948). (Russian)

The author considers the type problem for certain classes of simply connected Riemann surfaces which have on each sheet only two or three simple branch points lying over the finite portion of the $w$-plane. Use is made of quasi-conformal mapping and of Ahlfors' well-known criterion for parabolic type. Among various classes of surfaces considered is the following. Each surface of the class is determined by an infinite system of segments $\Delta_n$, $n=0, \pm 1, \pm 2, \cdots$, lying on the real axis $\Im w=0$ and possessing the property that two consecutive segments have alternately coincident left and right end points. To each segment $\Delta_n$ is made to correspond, as a sheet of the surface, a $w$-plane cut along the real axis from the end points of $\Delta_n$ to $\pm\infty$. The surface is obtained by connecting two consecutive sheets along the slit which they have in common in the manner of the surface for the square root. It is shown that every surface of this class is of parabolic type. Sufficient conditions for both parabolic and hyperbolic type are obtained for various classes of surfaces similar to that described above. Some of these results overlap in part with earlier ones obtained by Thiem Le-Van [Comment. Math. Helv. 20, 270–287 (1947); these Rev. 9, 139].

*W. Seidel* (Los Angeles, Calif.).

**Volkovyskiĭ, L. I. Investigations on the problem of type for a simply-connected Riemann surface.** Uspehi Matem. Nauk (N.S.) **3**, no. 3(25), 215–216 (1948). (Russian)

A summary, without specific theorems and proofs, is given of the author's thesis [Steklov Mathematical Institute]. Using methods of Ahlfors, Grötzsch, Lavrentieff and Kobayashi, he obtains sufficient conditions for simply connected Riemann surfaces to be of parabolic or hyperbolic type. [Cf. the preceding review.] *W. Seidel.*

**Volkovyskiĭ, L. I. Convergent sequences of Riemann surfaces.** Mat. Sbornik N.S. **23**(65), 361–382 (1948). (Russian)

Questions are studied which are related to the well-known kernel theorems of Carathéodory [Math. Ann. **72**, 107–144 (1912)]. A definition of the kernel of a sequence of Riemann surfaces is used which differs somewhat from that of Carathéodory. A sequence $\{F_n\}$ of arbitrary Riemann surfaces being given, the surfaces are said to have a common circle $Q_0: |w-a|<\rho$ if there exists on every surface $F_n$ a single-sheeted circle lying over $Q_0$. The kernel of $\{F_n\}$ with a common circle $Q_0$ is then defined to be the largest Riemann surface $F$ containing $Q_0$ with the following property: in every closed region $\Sigma$, lying wholly in $F$, there are at most a finite number of points such that, on removing from $\Sigma$ these points together with their neighborhoods of arbitrarily small radius $\delta$ (for the point at infinity such a neighborhood is taken to be the region $|w|>1/\delta$), the remaining part $\Sigma_\delta$ belongs to all surfaces $F_n$ from a certain value of $n$ on, which value will depend on $\Sigma$ and $\delta$. Points of $F$ for which no neighborhood exists belonging to all $F_n$ for all sufficiently large values of $n$ are called exceptional points of the kernel $F$ and of the sequence $\{F_n\}$. If $R$ is an arbitrary Riemann surface over the $z$-plane, $G_n$ an arbitrary sequence of regions in $R$, and $w=f_n(z)$ an arbitrary single-valued analytic function in $G_n$ which maps $G_n$ on a Riemann surface $F_n$, conditions are given which are necessary and, in some cases, sufficient for the convergence of $F_n$ to its kernel. Another type of theorem is as follows. Let there be given an arbitrary sequence of Riemann surfaces $\{F_n\}$ with a common circle $Q_0$ and a sequence of functions $z=\varphi_n(w)$, each defined, single-valued, and analytic on the corresponding $F_n$, mapping them on regions $G_n$ lying all in one and the same Riemann surface $R$ over the $z$-plane. Let $F$ be the kernel of the sequence $\{F_n\}$ defined by $Q_0$. If the sequence $\{\varphi_n(w)\}$ converges uniformly in a neighborhood of at least one point $w_0$ of $F$ and the limit function $\varphi(w)$ is not identically constant, then the sequence $\{\varphi_n(w)\}$ converges uniformly in every closed subregion of $F$ which contains no exceptional points. Carathéodory's kernel theorems, as well as a theorem of Elfving [C. R. Huitième Congrès Math. Scandinaves 1934, pp. 96–105 (1935)] are obtained by specialization of the author's theorems. Applications are made to the problem of type and to the compactness of various classes of Riemann surfaces. *W. Seidel* (Los Angeles, Calif.).

**Sario, Leo. Über Riemannsche Flächen mit hebbarem Rand.** Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **50**, 79 pp. (1948).

This paper is concerned with the concept of the removability (Hebbarkeit) of the ideal boundary of an abstract Riemann surface. The boundary of a Riemann surface is said to be removable provided that there exists no nontrivial analytic function on the surface with finite Dirichlet integral. It is easily concluded from a theorem of R. Nevanlinna that, if the boundary of a Riemann surface is null (in the sense of harmonic measure), then it is removable. The author gives a sufficient condition for the removability of the boundary of a Riemann surface and applies it and related criteria to a number of problems. The theorem in question may be stated as follows. Let $F$ denote an open Riemann surface and $\{F_n\}$ an exhaustion of $F$ by compact subregions. Only exhaustions meeting certain simple conditions are admitted. With each component $E_{ni}$ of $F_n - F_{n-1}$ is associated a number (greater than 1), its module, defined in terms of the harmonic measure with respect to $E_{ni}$ of the part of the boundary of $E_{ni}$ in the boundary of $F_n$. Let $\mu_n$ denote the minimum of the modules associated with the $E_{ni}$. Then $\prod_{n=1}^{\infty}\mu_n=+\infty$ implies that the boundary of $F$ is removable. It is shown that a linear Cantor set of linear measure zero is removable. Since a known result of R. Nevanlinna implies the existence of Cantor sets of linear measure zero but of positive harmonic measure, it is seen that the concept of removability is less restrictive than that of zero harmonic measure.

Applications are given to covering surfaces of the complex plane ramified over only a finite number of points and to unramified covering surfaces of abstract Riemann surfaces. In particular, it is shown that the covering surfaces of a closed Riemann surface associated with the Abelian integrals of the first kind have removable boundaries. A number of related questions are also considered (applications to uniformization theory, the continuation problem for Riemann surfaces, the mapping of plane regions on "Kreisgebiete"). *M. Heins* (Providence, R. I.).

**Petersson, Hans. Automorphe Formen als metrische Invarianten. I. Automorphe Formen, metrische Verknüpfung, Eigenfunktionen linearer Funktionale.** Math. Nachr. **1**, 158–212 (1948).

In der Arbeit, die aus zwei Teilen besteht, wird eine Theorie der durch ganze automorphe Formen uniformisierbaren multiplikativen und Abelschen Differentiale entwickelt, indem diese als metrische Invarianten von der Dimension $-2$ aufgefasst und als Grenzwerte metrischer Invarianten von der stetig veränderlichen reellen Dimension $-r$ für $r\to 2+0$ dargestellt werden. In dem vorliegenden ersten Teil wird der erforderliche Apparat geschaffen, indem eine analytische Formentheorie für komplexe Dimension und eine in wesentlichen Zügen neue Theorie der metrischen Invarianten reeller Dimension entwickelt wird.

*P. J. Myrberg* (Helsinki).

**Tsuji, Masatsugu. Some metrical theorems on Fuchsian groups.** Jap. J. Math. **19**, 483–516 (1947).

Extensions of known theorems concerning Fuchsian groups and capacity. *M. Heins* (Providence, R. I.).

**Zahorski, Z. On a problem of G. Choquet.** Časopis Pěst. Mat. Fys. **73**, 69–77 (1948). (English. Czech summary)

Choquet's problem is the following. Let $C$ be a continuum in the complex plane $\Pi$ of $z$ such that every connected component of $\Pi - C$ has a circumference for frontier and such that any two such circumferences are disjoint; let $Z=f(z)$ be a continuous complex-valued function on $C$, such that $f^{(1)}(z)=\lim(\Delta f/\Delta z)$ (as $\Delta z\to 0$) exists on $C$; $f(z)$ is termed monogenic on $C$. It is said that $C$ is an essential continuum if every function monogenic on $C$ is indefinitely derivable on $C$. Questions: (1) how to characterize essential continua; (2) if $C$ is an essential continuum, is it true that

the class of all functions monogenic on $C$ is quasianalytic (in the sense that if a function monogenic on $C$ is zero on a strict subcontinuum of $C$ it is identically 0)? The author proves that the only continuum which is an essential continuum is $\Pi$ itself; accordingly, the answer to (2) is "yes."

*W. J. Trjitzinsky* (Urbana, Ill.).

**Bochner, Salomon, and Martin, William Ted. Several Complex Variables.** Princeton Mathematical Series, vol. 10. Princeton University Press, Princeton, N. J., 1948. ix+216 pp. $4.00.

Das vorliegende Buch macht mit einer Reihe wichtiger Tatsachen der Funktionentheorie mehrerer komplexer Veränderlichen bekannt, wobei nach Absicht der Verf. Wiederholungen mit älteren Werken [Bergman, Behnke-Thullen und Osgood] nach Möglichkeit vermieden werden. Verf. streben in der Formulierung der Sätze nach möglichster Allgemeinheit, was auch dadurch zum Ausdruck kommt, dass viele Aussagen sich auf Funktionen beliebiger, reeller oder komplexer, Veränderlichen und andere auf rein formale Potenzreihen beziehen. Aus dem Inhalt werde folgendes erwähnt (ohne dass unsere Aufzählung vollständig wäre).

(I) Groups of transformations by formal power-series: Definition formaler Transformationen mittels formaler Potenzreihen:

$$(\mathfrak{T})\qquad f_\lambda(x)=\sum_{n_1,\cdots,n_k=1}^{\infty} \overset{\lambda}{a}_{n_1\cdots n_k}x^{n_1}\cdots x^{n_k},\qquad \lambda=1,\cdots,l;\ \overset{\lambda}{a}_{0\cdots 0}=0.$$

Falls $l=k$, wird $\mathfrak{T}$ eine "innere" Transformation des $(x)$-Raumes genannt. Übertragung der Sätze von Cartan und Carathéodory über analytische Abbildungen und Abbildungsfolgen auf formale innere Transformationen; Verallgemeinerungen [Behnke-Peschl] und Folgerungen dieser Sätze.

(II) Basic facts about analytic functions of real and complex variables: Definition einer analytischen Funktion reeller oder komplexer Veränderlichen durch konvergente Potenzreihen. Cauchy-Integral bei komplexen Veränderlichen und unmittelbare Folgerungen. Allgemeine Aussagen über die Fortsetzung komplex analytischer Funktionen, über die reell-analytischer Funktionen in solche komplexer Veränderlichen und über die Fortsetzung analytischer Funktionen gemischter Veränderlichen. Ein Satz über implizite Funktionen, die durch analytische Funktionen (reeller oder komplexer Veränderlichen) gegeben sind.

(III) Analytic mappings with a fixed point: Satz von Fatou-Bieberbach über die Abbildung des ganzen Raumes in einen echten Teil. Konkrete Anwendungen der in (I) entwickelten Sätze auf analytische Abbildungen im Raume der Veränderlichen $(z_1,\cdots,z_k)$. Übertragung des Schwarz'schen Lemmas und des Hadamard'schen Drei-Kreise-Satzes auf Systeme analytischer Funktionen $f_\lambda(z_1,\cdots,z_k)$, $\lambda=1,\cdots,n$.

(IV) Analytic completion: $\bar{D}$ heisse eine analytische Ergänzung des Bereiches $D\subset\bar{D}$, falls alle in $D$ analytischen Funktionen noch in $\bar{D}$ analytisch sind. Es werden reelle Veränderliche zugelassen, doch verlangt, dass mindestens eine Veränderliche komplex sei. Im Mittelpunkt steht der verallgemeinerte "Kontinuitätssatz." Folgerungen, z.B. der "Kugelsatz" und Verallgemeinerungen. Anwendungen auf sternartige und kreissymmetrische Bereiche.

(V) Singularities at boundary points: Definition der "Randeigenschaft" gewisser Randpunkte eines Bereiches $D$ im Raume $E_{2k}$ der komplexen $z_1,\cdots,z_k$. Zu einer Menge solcher Randpunkte $\{P\}$ gibt es stets eine in $D$ analytische, in der Umgebung von $\{P\}$ unbeschränkte Funktion. Hauptsatz über die gleichzeitige Fortsetzbarkeit analytischer Funktionen [Cartan-Thullen] und Folgerungen. Untersuchung der analytischen Ergänzungen eines "Tubus":

$$(T)\qquad [x_j\varepsilon S;\ -\infty<y_j<\infty],\qquad z_j=x_j+iy_j;\ j=1,\cdots,k,$$

wobei $S$ eine Punktmenge des $(x)$-Raumes ist. Ist $T$ ein Bereich, so ist die zugehörige konvexe Hülle die grösste analytische Ergänzung von $T$ (d.h. Regularitätshülle). (Weitere Aussagen über analytische Funktionen in einer Punktmenge $T$ in VI.)

(VI) Inequalities, bounds, and norms: Es seien einige Tatsachen herausgegriffen: $L_p$ $(p\geqq 1)$ sei die Menge der in einem Bereiche $D$ analytischen Funktionen $f(z_1,\cdots,z_k)$, für welche $\|f\|_p=(\int_D|f|^p dv_x dv_y)^{1/p}$ endlich ist (Verallgemeinerung der quadratintegrierbaren Funktionen, $p=2$). Nachweis der Existenz von "Minimalfunktionen" mit den bekannten Invarianzeigenschaften, die zur Aufstellung von Bereichklassen und Repräsentantenbereichen führen [S. Bergman]. Für $p=2$ ist der Kern des dann in $D$ existierenden vollständigen Orthogonalsystems nur von $D$ abhängig; es interessiert dessen Verhalten am Rande, das vermutlich mit der Möglichkeit analytischer Ergänzungen von $D$ zusammenhängt.

(VII) The theory of Hartogs. Subharmonic functions: Hauptziel des ersten Teiles ist der Hartogs'sche Satz, nach dem eine Funktion $f(z_1,\cdots,z_k)$, die in jeder einzelnen Veränderlichen bei Festhalten der übrigen analytisch ist, schlechthin analytisch ist. Im zweiten Teil Untersuchung der "Radien" (Regularitätsradien) $\sigma(z)=\sigma(z_1,\cdots,z_k)$ eines "extremen" Hartogs'schen Bereiches $D$ des $(w;z_1,\cdots,z_k)$-Raumes und Nachweis, dass $\log 1/\sigma(z)$ eine Hartogs'sche Funktion ist (Verallgemeinerung der subharmonischen Funktion für $k\geqq 2$). Die gleiche Eigenschaft gilt für Randfunktionen anderer, den Hartogs'schen verwandter Bereiche. Satz über komplexe Liesche Gruppen.

(VIII) Removable singularities: Begriff einer "verallgemeinerten" (nicht notwendig differenzierbaren) Lösung $f(x_1,\cdots,x_k)$ einer linearen partiellen Differentialgleichung $\Lambda f=0$, sodass $\int_D f(\Lambda^*\varphi)dv_x=0$ für jede "testing"-Funktion $\varphi$ ($\Lambda^*$ der zugeordnete Operator). Analog wird die verallgemeinerte Lösung eines Systems von Differentialgleichungen definiert (Beispiel: Cauchy-Riemannschen Differentialgleichungen). Bedingungen, unter denen ein verallgemeinertes, in einem Bereiche $D$ bis auf eine "Ausnahmemenge" $A$ gegebenes Lösungssystem $(f_1,\cdots,f_s)$ noch in ganz $D$ gültig ist, indem man die $f_\lambda$ in $A$ gleich Null setzt. Hauptsatz (und seine Verallgemeinerung), der besagt, dass eine Funktion $\psi(z_1,\cdots,z_k)$, die in einer vollen Umgebung $U$ eines Punktes $P$ stetig oder beschränkt ist, dazu analytisch in $U$ mit Ausnahme höchstens eines analytischen Flächenstückes $\Phi(z)=0$, $\Phi(0)=0$, in ganz $U$ analytisch ist. Satz über das Nichtverschwinden der Funktionaldeterminante einer topologischen, analytischen Transformation; und Folgerungen.

(IX) Algebraic theorems: Weierstrass'scher Vorbereitungssatz (für analytische Funktionen und formale Potenzreihen) und die dadurch gegebene Möglichkeit, eine analytische Funktion (Potenzreihe) mit Hilfe "ausgezeichneter (Pseudo-)Polynome" in einem gegebenen Punkte in irreduzible Faktoren zu zerlegen. Begriff der "charakteristischen" Mannigfaltigkeiten. Sätze über algebraische und rationale Funktionen, wobei aus dem algebraischen (rationalen) Verhalten nach Gruppen von Veränderlichen auf das algebraische (rationale) Verhalten schlechthin geschlossen wird.

(X) Local analytic varieties [nach W. Rückert]. Allgemeine analytische Gebilde $M$, d.h. gemeinsame Nullstellen eines Systems analytischer Funktionen. Jedem $M$ wird das Ideal aller auf $M$ verschwindenden, im gegebenen Bereich analytischen Funktionen zugeordnet. So wird die Untersuchung von $M$ und seine Zerlegung in irreduzible Mannigfaltigkeiten auf die obiger Ideale und ihre Zerlegung in Primideale zurückgeführt. Parameterdarstellung für irreduzible Mannigfaltigkeiten. *P. Thullen* (Panamá).

**Rothstein, Wolfgang. Die invariante Fassung des Kontinuitätssatzes für meromorphe Funktionen.** Arch. Math. 1, 119–126 (1948).

Die Behnkesche Fassung des Kontinuitätssatzes für reguläre Funktionen $f(z_1, \cdots, z_k)$, in welcher die analytischen Ebenen durch analytische Flächen ersetzt werden und die unter einschränkenden Bedingungen auch für meromorphe Funktionen gilt, wird in einer allgemeinen Form und ohne Einschränkung auf meromorphe Funktionen übertragen. Verf. beweist: $\mathfrak{F}_m$ mit den Rändern $\mathfrak{C}_m$ seien abgeschlossene Teile $2k$-dimensionaler irreduzibler analytischer Flächen $(k \leqq n-1)$, $\mathfrak{F}$ und $\mathfrak{C}$ seien abgeschlossene Punktmengen, sodass $\mathfrak{F}_m \to \mathfrak{F}$ und $\mathfrak{C}_m \rightrightarrows \mathfrak{C}$ (bei geeigneter Definition der Konvergenz), ferner sei die Funktion $g(z_1, \cdots, z_k)$ eindeutig und meromorph auf den $\mathfrak{F}_m$ und auf $\mathfrak{C}$; dann lässt sich $g$ eindeutig und meromorph nach $\mathfrak{F}$ fortsetzten. Es wird dabei nicht vorausgesetzt, dass die Grenzmannigfaltigkeit $\mathfrak{F}$ analytisch sei. *P. Thullen* (Panamá).

**Rothstein, Wolfgang. Die Existenz irreduzibler analytischer Flächen, welche sich über den Rand eines gegebenen Regularitätsbereiches nicht fortsetzen lassen.** Arch. Math. 1, 205–211 (1948).

Von der zwischen den wesentlichen Singularitäten analytischer Funktionen und analytischer Flächen bestehenden Analogie ausgehend stellt sich Verf. die Frage, ob es zu einem vorgegebenen Regularitätsbereich $\mathfrak{B}$ des Raumes $R_{2n}$ der komplexen Veränderlichen $z_1, \cdots, z_n$ stets eine in $\mathfrak{B}$ sich algebraisch verhaltende $(2n-2)$-dimensionale Fläche gibt, die sich nirgends über den Rand von $\mathfrak{B}$ hinaus fortsetzen lässt. Die Existenz derartiger Flächen in jedem schlichten und beschränkten Regularitätsbereich ergibt sich fast unmittelbar aus dem folgenden Satz: Gegeben seien ein solcher Bereich $\mathfrak{B}$ und eine in $\mathfrak{B}$ reguläre und eindeutige Funktion $g(z_1, \cdots, z_n)$. Erfüllen dann diejenigen Werte $a$ ($g=a$ in mindestens einem Punkte in $\mathfrak{B}$), für welche wenigstens eine irreduzible Komponente der Fläche: $g-a=0$ über den Rand von $\mathfrak{B}$ hinaus fortsetzbar ist, eine volle Kreisscheibe $K$, so ist auch die Funktion $g$ über $\mathfrak{B}$ hinaus fortsetzbar. Der Satz gilt noch, falls nur die Menge $K$ der "Ausnahmewerte" $a$ von positiver Kapazität ist. *P. Thullen* (Panamá).

## *Theory of Series*

**Copsey, E. H., Frazer, H., and Sawyer, W. W. A research project.** Math. Gaz. 32, iii–iv (1948).

The authors give further numerical data connected with Hilbert's inequality, extending results previously reported [Nature 161, 361 (1948); these Rev. 9, 344] to the range $n=0(1)19$. *R. P. Boas, Jr.* (Providence, R. I.).

**Erdös, P., Feller, W., and Pollard, H. A property of power series with positive coefficients.** Bull. Amer. Math. Soc. 55, 201–204 (1949).

Let $\sum_0^\infty u_n x^n = 1/(1-P(x))$, where $P(x)=\sum_0^\infty p_n x^n$ with $p_n \geqq 0$ and $\sum_0^\infty p_n = 1$. Suppose that $P(x)$ is not a series in $x^m$ for any $m=2, 3, \cdots$. Then the sequence $\{u_n\}$ converges to $\lim_{n\to\infty} (\sum_1^n k p_k)^{-1}$. Two proofs are offered, the first by way of Wiener's theorem on the reciprocal of an absolutely convergent Fourier series, and the second by an elementary computation with the difference equation for $\{u_n\}$. [In the author's statement of the theorem, $P(x)$ should be replaced by $P(x)-1$.] *R. C. Buck* (Providence, R. I.).

**Lorentz, G. G. A contribution to the theory of divergent sequences.** Acta Math. 80, 167–190 (1948).

Banach, in chapter 2 of his book [Théorie des Opérations Linéaires, Warszawa, 1932], defined generalized limits $L(x_n)$ applying to all bounded sequences $x_n$ in such a way that four useful conditions are satisfied. Let $x_n$ belong to class $U$ if $L(x_n)$ has the same value for all the different $L$'s satisfying Banach's conditions. The class $U$ contains, among others, all convergent sequences, all periodic sequences, and all bounded gap sequences, that is, sequences in which $x_n=0$ except when $n=n_1, n_2, \cdots$, where $n_k - n_{k-1} \to \infty$. A regular matrix transformation evaluates each sequence $x_n$ in $U$ to $L(x_n)$ if and only if

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| = 0.$$

Many regular matrix methods of the Hurwitz-Silverman-Hausdorff type, including those of Cesàro and Euler, satisfy this condition. If a regular matrix method evaluates all sequences in $U$, it must also evaluate some bounded sequences not in $U$. There are four theorems on gap sequences, of which the following is the simplest. In order that a regular matrix method $a_{nk}$ be such that, whenever $n_1, n_2, n_3, \cdots$ is a sequence of integers increasing sufficiently rapidly, each sequence $x_n$ for which $x_n=0$ when $n \neq n_1, n_2, n_3, \cdots$ is evaluable by the method, it is necessary and sufficient that $\lim_{n\to\infty} \max_{k=0,1,2,\ldots} |a_{nk}| = 0$. *R. P. Agnew* (Ithaca, N. Y.).

**Basu, S. K. On the total regularity of some integral and sequence transformations.** J. London Math. Soc. 23, 300–309 (1948).

Let $K(x, t)$ be, for each $x>0$, Lebesgue integrable over $0 \leqq t \leqq x$ so that each function $s(t)$ bounded and measurable over each finite interval has a transform $\sigma(x)$ defined by $\sigma(x) = \int_0^x K(x, t)s(t)dt$, $x>0$. This transformation is regular if $\lim_{x\to\infty} \sigma(x) = \lim_{t\to\infty} s(t)$ whenever $\lim s(t)$ exists and is finite, and is totally regular if it is regular and also $\lim \sigma(x) = +\infty$ whenever $\lim s(t) = +\infty$. The following theorem is analogous to a theorem of W. A. Hurwitz [Proc. London Math. Soc. (2) 26, 231–248 (1927)] on matrix transformations. The above function-to-function kernel transformation is totally regular if and only if it is regular (conditions for this are known) and there is a number $T \geqq 0$ such that, for each $x>t$, $K(x, t) \geqq 0$ for almost all $t$ in the interval $T \leqq t \leqq x$. This result is used to obtain theorems of Mercerian type which involve infinite limits. *R. P. Agnew* (Ithaca, N. Y.).

**Basu, S. K. On the total relative strength of the Hölder and Cesàro methods.** Proc. London Math. Soc. (2) **50**, 447–462 (1949).

Let $C_n$ and $H_n$ denote the Cesàro and Hölder transforms, of the same order $\alpha > -1$, of a real sequence $s_n$. A classic equivalence theorem asserts that $\lim C_n = \lim H_n$ whenever one of the limits is finite. Six theorems augment previously known facts when one of the limits is $+\infty$. If $-1<\alpha<0$, then $C_n \to +\infty$ implies $H_n \to +\infty$; but the converse is false. If $0<\alpha<1$, then $H_n \to +\infty$ implies $C_n \to +\infty$; but the converse is false. If $\alpha>1$, then $C_n \to +\infty$ implies $H_n \to +\infty$; but the converse is false. The remaining theorems involve Cesàro methods of one order and Hölder methods of another order. The proofs involve the theory of transformations of the type introduced by W. A. Hurwitz and Silverman [Trans. Amer. Math. Soc. **18**, 1–20 (1917)] and treated further by Hausdorff [Math. Z. **9**, 74–109, 280–299 (1921)]. Use is made of the following lemma. Two transformations of the Hurwitz-Silverman-Hausdorff type generated by sequences of nonzero elements cannot be totally equivalent (that is, equivalent for both finite and infinite limits) unless they are identical. According to the author, Bosanquet gave this lemma in class lectures on divergent series.

*R. P. Agnew* (Ithaca, N. Y.).

**Kuttner, B. A theorem on Hölder means.** J. London Math. Soc. **23**, 315–320 (1948).

It is shown that if $r>1$ ($r$ not necessarily an integer) then the Euler-Abel power series method of summability does not totally include the Hölder method $H_r$ of order $r$. Bosanquet [same J. **21**, 11–15 (1946); these Rev. **8**, 259] obtained the result for integers $r>1$. Actually, it is shown that, if $r>1$, then there is a real series $\sum u_k$ which is evaluable $H_r$ to $+\infty$ and which is such that $\sum t^k u_k$ converges when $0<t<1$ and defines a power series transform $\sigma(t)$ which fails to be such that $\sigma(t)\to\infty$ as $t\to 1$.

*R. P. Agnew* (Ithaca, N. Y.).

**Bosanquet, L. S. Note on convergence and summability factors. III.** Proc. London Math. Soc. (2) **50**, 482–496 (1949).

[For part II cf. the same vol., 295–304 (1948); these Rev. **10**, 112.] This paper gives a comprehensive unification and generalization of many known theorems which, for restricted sets of values of $\alpha$ and $\beta$, give conditions on a sequence $\epsilon_0, \epsilon_1, \cdots$ necessary or sufficient, or both, to ensure that the series $\sum \epsilon_n a_n$ is summable by the Cesàro method $(C, \beta)$ whenever $\sum a_n$ is summable $(C, \alpha)$. In the following main result, the numbers, $\alpha$, $\beta$, and $p$ are not necessarily integers and the relation $S_n^\alpha = o(n^\alpha)$ means as usual that $\sum a_n$ is summable $(C, \alpha)$ to 0. If $0\leqq\beta\leqq\alpha$, $p\geqq 0$, then necessary and sufficient conditions that $\sum \epsilon_n a_n$ should be summable $(C,\beta)$ whenever $S_n^\alpha = o(n^{\alpha+p})$ are (1) $\epsilon_n = O(n^{\beta-\alpha-p})$, (2) $\sum n^{\alpha+p}|\Delta^{\alpha+1}\epsilon_n| < \infty$. If $0\leqq\alpha<\beta$, $p\geqq 0$, the conditions are (1′) $\epsilon_n = O(n^{-p})$ and (2). The same holds if $O$ and $o$ are interchanged throughout.

*R. P. Agnew* (Ithaca, N. Y.).

**Teghem, J. Une généralisation des théorèmes $E\to B$ et $B\to E$ de M. Knopp.** Nieuw Arch. Wiskunde (2) **23**, 8–12 (1949).

Methods $(B, \theta, m)$ of summability, modifications of the Borel method which depend upon an angle $\theta$ and a nonnegative integer $m$, are defined. Then methods $(E, b, m)$, modifications of the Euler-Knopp method which depend upon a complex number $b$ and a nonnegative integer $m$, are defined. For these methods, the author proves Abelian and Tauberian theorems analogous to the Abelian and Tauberian theorems proved by Knopp [Math. Z. **18**, 125–156 (1923)] to show the relative effectivenesses of the Borel and Euler-Knopp methods.

*R. P. Agnew* (Ithaca, N. Y.).

**Delange, Hubert. The converse of Abel's theorem on power series.** Ann. of Math. (2) **50**, 94–109 (1949).

The author gives proofs of results stated in a previous paper [C. R. Acad. Sci. Paris **224**, 436–438 (1947); these Rev. **8**, 457], together with some further refinements. He includes a simple new proof of Littlewood's theorem on the converse of Abel's theorem.

*H. R. Pitt* (Belfast).

**Hsu, Leetch C. Approximations to a class of double integrals of functions of large numbers.** Amer. J. Math. **70**, 698–708 (1948).

Let $I_n$ be the double integral of $\varphi(x, y)e^{nh(x,y)}$, taken over a two-dimensional closed region $S$ which contains the origin; let $\varphi$ be absolutely integrable over $S$; suppose that to any positive $p$ a positive number $\epsilon$ corresponds such that $h(x, y)<h(0, 0)-\epsilon$ for every point $(x, y)$ of $S$ which has distance greater than $p$ from the origin. Under very general conditions concerning the first and second derivatives of $h(x, y)$ at the origin, the author proves that $ne^{-nh(0,0)}I_n$ tends to $2\pi\Delta^{-\frac{1}{2}}\varphi(0, 0)$, if $n\to\infty$. Here $\Delta$ denotes the positive value of the function $h_{xx}h_{yy}-h_{xy}^2$ at the origin. If an analytic curve passing through the origin divides $S$ into two subregions with the contributions $I_n'$ and $I_n''$, then $ne^{-nh(0,0)}I_n'$ and $ne^{-nh(0,0)}I_n''$ tend to $\pi\Delta^{-\frac{1}{2}}\varphi(0, 0)$. If the origin is a boundary point of $S$ he proves under very general conditions that $n^{\frac{3}{2}}e^{-nh(0,0)}I_n$ tends to $(2\pi)^{\frac{1}{2}}\varphi(0, 0)\,(h_x^2+h_y^2)^{-\frac{1}{2}}(K-K')^{-1}$. Here $K$ and $K'$ denote the curvatures at the origin of the boundary of $S$ and of the curve $h(x, y)=h(0, 0)$.

*J. G. van der Corput* (Amsterdam).

## *Fourier Series and Generalizations, Integral Transforms*

**Alaci, V. Sur une intégrale définie necessaire dans l'étude de séries trigonométriques.** Bull. Sci. Tech. Polytech. Timişoara **13**, 121–127 (1948).

The known value of $\int_0^\infty t^{-q}\sin^p t\,dt$, with $q/p$ rational, $0\leqq q/p\leqq 1$, is obtained simply after repeated integration by parts and use of the Fourier sum of $\sin^p t$.

*P. Civin.*

**Geronimus, Ya. L. Refinement of estimates of van der Corput, Visser, Fejes and Boas for the coefficients of trigonometric polynomials.** Doklady Akad. Nauk SSSR (N.S.) **63**, 479–482 (1948). (Russian)

The estimates in question involve a real trigonometric polynomial $F(t)=\sum_{-n}^{n} a_j e^{ijt}$ and are of the form

$$\lambda_0|a_0|+\lambda_k|a_k|\leqq C\int_0^{2\pi}|F(t)|\,dt.$$

For references to recent work see Boas [Nederl. Akad. Wetensch., Proc. **50**, 759–762 = Indagationes Math. **9**, 369–372 (1947); these Rev. **9**, 345]. The author points out that the best possible estimates are obtainable from more general previous results of his own [C. R. Acad. Sci. Paris **198**, 2221–2222 (1934); **199**, 1010–1012 (1934)]; the authors cited in the title gave results which are best possible only for $k>\frac{1}{2}n$ ($\lambda_0\neq 0$) or $k>\frac{1}{3}n$ ($\lambda_0=0$). For example, the best $C$

in $|a_k| \leqq C\int_0^{2\pi} |F(t)|dt$ is $\frac{1}{4}\nu^{-1}$, where $\nu$ is the least positive root of

$$\begin{vmatrix} 2 & (2\nu)/1! & \cdots & (2\nu)^p/p! \\ (2\nu)/1! & 2 & \cdots & (2\nu)^{p-1}/(p-1)! \\ & \cdots & \cdots & \\ (2\nu)^p/p! & (2\nu)^{p-1}/(p-1)! & \cdots & 2 \end{vmatrix} = 0,$$

and $p = [n/k]$. *R. P. Boas, Jr.* (Providence, R. I.).

**Siddiqi, J. A. On the harmonic summability of Fourier series.** Proc. Indian Acad. Sci., Sect. A. **28**, 527–531 (1948).

Following M. Riesz [Proc. London Math. Soc. (2) **22**, 412–419 (1924)] a sequence $s_0, s_1, \cdots$ is summable by the harmonic mean to limit $s$ if $(\log n)^{-1}\sum_{k=0}^{n} s_{n-k}/(k+1) \rightarrow s$ as $n\rightarrow\infty$. The author shows that the Fourier series of a function $f$ is summable to $f(x)$ at a point $x$ where $\int_0^t g(u)du = o(-t/\log t)$, with $g(u) = f(x+u)+f(x-u)-2f(x)$ [see also Zygmund, C. R. Acad. Sci. Paris **179**, 870–872 (1924); Hille and Tamarkin, Trans. Amer. Math. Soc. **34**, 757–783 (1932); Iyengar, same Proc., Sect. A. **18**, 81–87, 113–120 (1943); these Rev. **5**, 65]. *A. Zygmund.*

**de Sz. Nagy, Béla. Sur une classe générale de procédés de sommation pour les séries de Fourier.** Hungarica Acta Math. **1**, no. 3, 14–52 (1948).

Let $\varphi(u)$ be defined for $0 \leqq u \leqq 1$. The author applies to certain problems of Fourier series the method of summability defining the sum of a series $u_0+u_1+u_2+\cdots$ as the limit for $n\rightarrow\infty$ of

$$(*) \qquad u_0 + \phi(1/n)u_1 + \phi(2/n)u_2 + \cdots + \phi((n-1)/n)u_{n-1}$$

[see also Perron, Math. Z. **6**, 286–310 (1920)]. Here are some of the results obtained. (1) Suppose that $\phi(u)$ is absolutely continuous in (0, 1) and that its derivative $\phi'(u)$ is of bounded variation except in the neighborhood of a finite number of points. Then the Lebesgue constants $\sigma_n$ corresponding to the method (*) satisfy the inequality

$$(\ddagger) \quad \sigma_n \leqq |\phi'(+0)| + 2\pi^{-1}\int_{+0}^{1-0}(1-u)\left\{2+\log\frac{1+u}{1-u}\right\}|d\phi'(u)|.$$

(2) If $\phi(u)$ is decreasing and convex, then $\sigma_n = 1$. (3) An inequality corresponding to ($\ddagger$) is established for the conjugate Lebesgue constants $\bar\sigma_n$ (Lebesgue constants for conjugate series). (4) Under the assumptions of (2), $\bar\sigma_n = 4\pi^{-1}\sum_{i=0}^{N}(2i+1)^{-1}\phi((2i+1)/n)$ $(N = [\frac{1}{2}(n-1)])$.

Let $\sigma_n(x;f)$ be the means (*) for the Fourier series of a function $f$, and let $\rho_n{}^\alpha$, $\rho_n{}^{(r)}$, $\rho_n{}^{(r)\alpha}$ denote the upper bounds in $0 \leqq x \leqq 2\pi$ of the expression $|f(x)-\sigma_n(x;f)|$ for all $f$ satisfying respectively the conditions (a) $|f(x+h)-f(x)| \leqq |h|^\alpha$, (b) $|f^{(r)}(x)| \leqq 1$, (c) $|f^{(r)}(x+h)-f^{(r)}(x)| \leqq |h|^\alpha$. Then (5) $\rho_n{}^\alpha = O(n^{-\alpha})$ if $\phi$ satisfies the assumptions of (1), the right side of ($\ddagger$) is finite and $\int|u-a|^{1-\alpha}|d\phi'(u)|$ is finite in the neighborhood of every exceptional point $a$ mentioned in (1). (6) Let $\psi_r(u) = u^{-r}(1-\phi(u))$, $r>0$, and suppose that it satisfies the conditions imposed on $\phi(u)$ in (1); then $\rho_n{}^{[r]} = O(n^{-r})$. (7) Under assumptions similar to those in (5) and (6), $\rho_n{}^{[r]\alpha} = O(n^{-r-\alpha})$. *A. Zygmund* (Chicago, Ill.).

**Chandrasekharan, K., and Szász, Otto. On Bessel summation.** Amer. J. Math. **70**, 709–729 (1948).

Let $\alpha_\mu(t) = J_\mu(t)/J_\mu(0)$ $(\mu+\frac{1}{2}>0)$, where $J_\mu(t)$ is Bessel's function of order $\mu$. A series $\sum a_n$ is said to be summable $J_\mu$ to $s$ if $\sum a_n\alpha_\mu(nt)$ converges for small $t$ and its sum tends to $s$ as $t\rightarrow 0$. For example, the $J_{\frac{1}{2}}$ sum is Riemann's limit: $\lim \sum a_n(nt)^{-1}\sin nt$.

The authors prove, among other results, the following. (1) If $0<\alpha<\delta<1$ and (i) $a_n = O(n^{-\delta})$, (ii) $\sum_n^\infty a_\nu = o(n^{-\alpha})$, then $\sum a_n$ is summable $J_\mu$ for $\mu+\frac{1}{2} = (1-\delta)/(1-\delta+\alpha)$. (2) If $\sum a_n$ is summable $(C, r+1)$ and strongly bounded $(C, r+1)$, then it is summable $J_\mu$ for $\mu+\frac{1}{2}>r+1>0$.

If $\sum a_n \cos nt$ is the Fourier series of a Lebesgue integrable function $\varphi(t)$, then (*) $\sum a_n\alpha_\mu(nt) = \varphi^*_{\mu+\frac{1}{2}}(t)$ $(\mu+\frac{1}{2}\geqq 1)$, where

$$\varphi^*_p(t) = \frac{2\Gamma(p+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(p)}t^{-2p+1}\int_0^t (t^2-u^2)^{p-1}\varphi(u)du$$

$(p>0)$. If $\varphi(t) = pt^{-p}\int_0^t(t-u)^{p-1}\varphi(u)du$ $(p>0)$, then the limit of $\varphi_p{}^*(t)$ as $t\rightarrow 0$ exists if and only if that of $\varphi_p(t)$ does. Thus, provided that (*) continues to hold for appropriate values of $\mu+\frac{1}{2}<1$, we have: (3) [and (4)]: if $\sum a_n\cos nt$ is the Fourier series of $\varphi(t)$, then the conclusion in (1) [and (2)] is that $\varphi_{\mu+\frac{1}{2}}(t)$ tends to a limit as $t\rightarrow 0$. Result (4) was given by the reviewer for $r=0$ [J. London Math. Soc. **7**, 47–52 (1932); cf. also Trans. Amer. Math. Soc. **39**, 189–204 (1936), theorem 6]; it generalises a result of Paley, Verblunsky and the reviewer which arose out of the work of Hardy and Littlewood [Paley, Proc. Cambridge Philos. Soc. **26**, 173–203 (1930); Verblunsky, ibid., 152–157 (1930); Bosanquet, Proc. London Math. Soc. (2) **31**, 144–164 (1930); Hardy and Littlewood, Math. Z. **19**, 67–96 (1923)].

To extend (*) for use in (3) and (4), the authors appeal to a method of the reviewer [loc. cit. (1930)]. But it seemed to the reviewer, after the publication of his paper, that his argument failed to establish the existence of $\varphi_{\mu+\frac{1}{2}}(t)$, and so of $\varphi^*_{\mu+\frac{1}{2}}(t)$, in a set of measure zero; and he added a supplementary argument [Proc. London Math. Soc. (2) **33**, 561 (1932)]. His second argument, suitably modified, establishes the existence of $\varphi_{\mu+\frac{1}{2}}(t)$ and $\varphi^*_{\mu+\frac{1}{2}}(t)$, but only as integrals of the form $\lim_{\epsilon\rightarrow 0}\int_0^{t-\epsilon}$. Once this point has been settled, the first method establishes (*) in the corresponding sense; and the proof of the equivalence of the limits of $\varphi_p{}^*(t)$ and $\varphi_p(t)$ holds with either interpretation.

*L. S. Bosanquet* (London).

**Bochner, S., and Chandrasekharan, K. Gauss summability of trigonometric integrals.** Amer. J. Math. **71**, 50–59 (1949).

In a previous paper [Proc. London Math. Soc. (2) **50**, 210–222 (1948); these Rev. **10**, 113] Chandrasekharan, inverting a formula of Bochner [Trans. Amer. Math. Soc. **40**, 175–207 (1936)] expressed the spherical averages of a function $f(x_1, \cdots, x_k)$ in terms of the typical $(\nu^2, \delta)$ means of the Fourier series of $f$. The inversion formula [see loc. cit.] had meaning for $\delta>\frac{1}{2}(k-1)$ only. It is now shown that if instead of ordinary convergence of the inverting integral we consider its Gauss summability, the formula remains valid (under very mild continuity conditions imposed on $f$) for all $\delta\geqq 0$. *A. Zygmund* (Chicago, Ill.).

**Zitarosa, Antonio. Una condizione sufficiente per i coefficienti di Legendre di una funzione.** Giorn. Mat. Battaglini (4) **2**(78), 3–9 (1948).

Suppose that $\{a_n\}$ is a sequence of numbers such that $\{a_n(2n+1)^{-\alpha}\}$ is of bounded variation for some $\alpha$, $0<\alpha<\frac{1}{2}$. Then the series $\sum a_nP_n(x)$ of Legendre polynomials converges uniformly in every interval interior to $(-1, 1)$ and is the Fourier series of an absolutely integrable function.

*A. Zygmund* (Chicago, Ill.).

**Agarwal, R. P. On some new kernels and functions self-reciprocal in the Hankel transform.** Proc. Nat. Inst. Sci. India 13, 305–318 (1947).

This is a continuation of a recent paper [Proc. Indian Acad. Sci., Sect. A. 27, 141–146 (1948); these Rev. 9, 582] and includes three similar theorems on pairs of functions whose resultant is a kernel transforming an $R_\mu$ into an $R_\nu$. The theorems may be used to derive new $R_\nu$ functions, and some examples involving sums of generalized hypergeometric functions are included. *M. C. Gray* (Murray Hill, N. J.).

**Parodi, Maurice. Sur une propriété des noyaux réciproques.** C. R. Acad. Sci. Paris 227, 810–812 (1948).

The function $K(t, x)$ is a reciprocal kernel, i.e.,

$$g(t)=\int_0^\infty K(t,x)f(x)dx$$

and

$$f(t)=\int_0^\infty K(t,x)g(x)dx$$

imply each other, if there exist functions $\Phi(p, t)$, $\rho(p)$ and $\psi(p)$ such that $\psi(\psi(p))=p$, $\rho(p)\rho(\psi(p))=1$, and

$$\int_0^\infty \Phi(p,t)K(t,x)dx=\rho(p)\Phi(\psi(p),x).$$

Three examples involving trigonometric and Bessel functions are given. The work is formal. *A. Erdélyi.*

**Parodi, Maurice. Nouvelles correspondances symboliques et leurs applications à la résolution d'équations intégrales.** Revue Sci. 86, 286–287 (1948).

Given the Laplace transform of $f(t)$, the author obtains the Laplace transforms of $f(t)\exp(-a^2t^{\pm n})$ ($n$ is a positive integer), and uses his result to solve certain integral equations. *A. Erdélyi* (Pasadena, Calif.).

**Ivanov, V. K. A generalized Fourier transform in operational calculus.** Mat. Sbornik N.S. 23(65), 383–398 (1948). (Russian)

The paper develops an operational calculus in which the difficulties due to nonconvergence of Fourier integrals are overcome for functions of polynomial order at infinity. Linear systems (vector spaces with complex scalars, not topologized) and their operators $A$, homomorphisms of a linear subsystem onto another linear subsystem, are discussed. The operator $A$ is called Hermitian if for all complex $\lambda$ the equation $(A-\lambda E)u=f$ has a solution $u=R(\lambda)f$ for all $f$ in the system $M$. It is shown, by a process similar to the extension of an integral domain to its quotient field, that for any linear system $M$ and Hermitian $A$ in $M$ there exists an extension $N$ of $M$ and an operator $\bar{A}$ with all $N$ as domain such that $\bar{A}$ is an extension of $A$. The elements of $N$ are the formal fractions $f/R^k$, where $k$ is any integer, $R=R(i)$, and are said to be in canonical form if $f\epsilon M$ but $f/R$ is not in $M$.

Here $M$ is taken to be the set of all functions such that $f(x)/(x-i)^k$ is in $L(-\infty, \infty)$ for some integer $k$, and $A$ is taken to be the operator $P=i\,d/dx$, while $Q$ denotes multiplication by $x$. Then $P$ and $Q$ are Hermitian in $M$, and the extension of $M$ for $P$ gives a set $N$ of quasifunctions in which every element is infinitely differentiable in the generalised sense.

If $T$ is the Fourier transform, defined in $L(-\infty, \infty)$, then the operator which, to an $f$ such that $f(x)/(x-i)^k$ is in $L(-\infty, \infty)$, associates the element $(P-iE)^kT(-Q-iE)^{-k}f$ of $N$ is an extension of $T$ to $M$, also called $T$. This generalisation of the Fourier transform is similar to that of Bochner [Vorlesungen über Fouriersche Integrale, Leipzig, 1932]. If $g=(P-iE)^kf$ is the canonical form of an element of $N$, the operator $Tg=(Q-iE)^kf$ is the extension of $T$ to $N$. The formal inverse $T'$ of $T$ is extended similarly, and the usual formulae of operational calculus, $TP=QT$, $T'P=-QT$, $TQ=-PT$, $T'Q=PT'$ and the Fourier inversion formula $TT'=E$ hold throughout $N$. The theory works similarly for several variables, and is illustrated by the case of two variables.

A method is given of using this calculus for solving equations in several variables. A function $\varphi(x)$ of $x=(x_1, \cdots, x_n)$ is called a multiplicator if it is infinitely differentiable in the normal sense and the product of any differential coefficient by any function of $M$ is in $M$. Elements which are finite linear combinations of elements of form $g\cdot\varphi(t, x)$, where $g$ is in $N$ and $\varphi(t, x)$ is a multiplicator for each $t$, form a linear system $N(t)$, in which convergence, differentiation with respect to $t$, etc., are defined by reference to term by term operation on the multiplicators in the finite sums. Partial differential equations in $x$ and $t$ can be solved, within the system $N(t)$, by Fourier transformation with respect to the $x$ variables. The wave equation in two dimensions is solved as an illustration. No discussion whether the solution thus found is a solution, or a function, in the ordinary sense is given, and uniqueness is not discussed.

The quasifunctions and generalised transforms of this paper are parallel to those of L. Schwartz [Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 21, 57–74 (1946); 23, 7–24 (1948); these Rev. 8, 264; 10, 36] but differ widely in definition. The operational calculus of both these theories is restricted to functions of polynomial order, whereas the Laplace transform or the Titchmarsh generalised Fourier transform [Introduction to the Theory of Fourier Integrals, Oxford, 1937] allow functions of exponential order to be dealt with. *J. L. B. Cooper* (London).

**Macfarlane, G. G. The application of Mellin transforms to the summation of slowly convergent series.** Philos. Mag. (7) 40, 188–197 (1949).

Let $F(s)$ be the Mellin transform of $f(x)$, convergent for $\sigma_1<\sigma<\sigma_2$, $s=\sigma+it$. Then

$$2\pi i\sum_{n=0}^\infty f(n+a)=\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)\zeta(s,a)ds,$$

where $\zeta(s,a)=\sum_{n=0}^\infty(a+n)^{-s}$. By deforming the contour in the integral on the right, the series on the left can sometimes be evaluated. The author gives three examples. One is an asymptotic series for $\sum_{m=1}^M(1-xm^{\frac{1}{2}})^{\frac{1}{2}}m^{-\frac{1}{2}}$, where $M=[x^{-\frac{1}{2}}]$; another is

$$\sum_{n=0}^\infty\frac{(-1)^n}{2n+1}J_1((2n+1)y)=\tfrac{1}{2}\sum_{n=0}^\infty\frac{(-1)^{n+1}B_{2n+1}(\frac{1}{2})(2y)^{2n+1}}{(2n+1)\Gamma(n+1)\Gamma(n+2)},$$

where the $B$'s are Bernoulli polynomials. To facilitate further applications the author also gives a table of 72 pairs of Mellin transforms. *R. P. Boas, Jr.* (Providence, R. I.).

**Sumner, D. B. An inversion formula for the generalized Stieltjes transform.** Bull. Amer. Math. Soc. 55, 174–183 (1949).

A complex inversion formula is obtained for the generalized Stieltjes transform $f(z)=\int_0^\infty\phi(t)(z+t)^{-\rho}dt$ ($\rho>0$). It is shown that if $\phi(t+)$ and $\phi(t-)$ both exist then

$\lim_{\eta\to 0+} M_\eta(f)=\frac{1}{2}[\phi(t+)+\phi(t-)]$, where

$$M_\eta(f)=(2\pi i)^{-1}\int_{C_{\eta t}}(s+t)^{s-1}f'(s)ds.$$

Here $C_\eta$ is the path which proceeds from $-t-i\eta$ along the straight line $\Im s=-\eta$ to $-i\eta$, then along the semicircle $|s|=\eta$, $\Re s\geqq 0$, to the point $i\eta$, and finally along the line $\Im s=\eta$ to $-t+i\eta$. *I. I. Hirschman, Jr.*

**Hirschman, I. I., Jr., and Widder, D. V. Generalized inversion formulas for convolution transforms.** Duke Math. J. 15, 659–696 (1948).

In a recent abstract [Proc. Nat. Acad. Sci. U. S. A. 34, 152–156 (1948); these Rev. 10, 36] the authors have announced an inversion theory for convolution transforms (*) $f(x)=\int_{-\infty}^{\infty}G(x-t)d\alpha(t)$ for which the bilateral Laplace transform $F(s)$ of $G(t)$ has the form

$$1/F(s)=e^{bs}\prod_1^\infty(1-s/a_k)e^{s/a_k}$$

and the equation (*) is inverted by use of the operator $1/F(D)$. In the present paper the authors modify the operator by the insertion of factors which tend to 1 in the limit; the explicit formula is too complicated to reproduce here.

The principal results are: (i) a proof that the new operator accomplishes the inversion of (*); (ii) an application of the new operator to obtain a complex inversion formula for (*); the only example given beyond the standard one, $F(s)=\sec\pi s$, is $F(s)=1/J_0(s)$ [the authors overlook the example $F(s)=\Gamma(a-s)\Gamma(a+s)$ which leads to kernels familiar in Tauberian theory]; (iii) the derivation of a sufficient condition that $f(x)$ have the form (*) with $\alpha(t)$ nondecreasing.

On page 693 theorems of Levinson are invoked to evaluate the number $\lim\omega(t)/t$, but it is not proved that his hypotheses are fulfilled. [This defect can be remedied by a lemma in Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford, 1937, p. 325]. No reference is given to the "well-known" theorem on linear differential operators used on page 668. Theorem VII seems to require the additional hypothesis that $\chi(s)$ be analytic. On page 694 the change of variables should read $s=e^{-x}$, $u=e^{-t}$. *H. Pollard.*

**Fox, C. Chain transforms.** J. London Math. Soc. 23, 229–235 (1948).

It is shown that for "chains" of the form

$$g_2(x)=\int_0^\infty g_1(u)k_1(ux)du,\ \cdots,\ g_n(x)=\int_0^\infty g_{n-1}(u)k_{n-1}(ux)du,$$

$$g_1(x)=\int_0^\infty g_n(u)k_n(ux)du,$$

there exists a theory similar to the standard theory for $n=2$. There are some differences between the cases of odd and even $n$, and these are illustrated with many examples.

*H. Pollard* (Ithaca, N. Y.).

**Beurling, Arne. On the spectral synthesis of bounded functions.** Acta Math. 81, 14 pp. (1948).

The spectrum $\Lambda_\varphi$ of a bounded function $\varphi(x)$ $(-\infty<x<\infty)$ is defined as the set of real numbers $\lambda$ for which $e^{i\lambda x}$ is contained in the manifold spanned by the set $\{\varphi(x+t)\}$ $(-\infty<t<\infty)$ in the weak topology of bounded functions, i.e., for every $G(x)\varepsilon L(-\infty,\infty)$ the condition $\int_{-\infty}^{\infty}\varphi(x+t)G(x)dx=0$ $(-\infty<t<\infty)$ implies $\int_{-\infty}^{\infty}e^{i\lambda x}G(x)dx=0$. The main problem of spectral synthesis is to decide whether, conversely, $\varphi(x)$ is contained in the weak closure of the set spanned by the functions $e^{i\lambda x}$, where $\lambda\varepsilon\Lambda_\varphi$. L. Schwartz has shown by an example [C. R. Acad. Sci. Paris 227, 424–426 (1948); these Rev. 10, 249] that for a bounded function in a Euclidean space of dimension $n\geqq 3$ the spectral synthesis is in general not possible in the weak topology. It seems likely that the weak topology is too narrow even in the one-dimensional case. In order to find more appropriate topologies, the author sets the following problem. For what positive weight-functions $w(x)\varepsilon L(-\infty,\infty)$ is it true that to any bounded measurable $\varphi(x)$ and to any $\epsilon>0$ there exists a trigonometric polynomial $\psi(x)=\sum c_\nu e^{i\lambda_\nu x}$ $(\lambda_\nu\varepsilon\Lambda_\varphi)$ such that $\int_{-\infty}^{\infty}|\varphi(x)-\psi(x)|w(x)dx<\epsilon$? He calls such a weight-function regular.

The function $g(t)$ is called a contraction of $f(t)$ if $|g(t_2)-g(t_1)|\leqq|f(t_2)-f(t_1)|$ for any pair of arguments $t_1$, $t_2$. Let $A$ denote the space of functions $f(t)=\int_{-\infty}^{\infty}e^{itx}F(x)dx$ $(F\varepsilon L)$ with the metric $\|f\|=\int_{-\infty}^{\infty}|F(x)|dx$. Let $A^*$ be the subset of $A$ consisting of the $f(x)$ for which also $F^*(x)=\sup_{|\xi|>|x|}|F(\xi)|\varepsilon L$; in $A^*$ use the metric

$$\|f\|^*=\int_{-\infty}^{\infty}|F^*(x)|dx.$$

The function $f\varepsilon A$ is called contractible in $A$ if each of its contractions $g$, normalized by the condition $\lim_{t\to\infty}g(t)=0$, also belongs to $A$. It is called uniformly contractible in $A$ if, for any sequence $\{g_n\}$ of normalized contractions of $f$, converging to 0 in the ordinary sense as $n\to\infty$, the norm $\|g_n\|$ also converges to 0.

It is shown that a sufficient condition for $w(x)$ to be regular is that every measurable $F(x)$ such that $|F(x)|\leqq w(x)$ have a Fourier transform uniformly contractible in $A$. It is then shown that, in particular, every $f\varepsilon A^*$ is uniformly contractible in $A$, and for every normalized contraction $g$ of $f(\neq 0)$ we have $\|g\|<5\|f\|^*$. From these results follows immediately the main result of the paper: if $w(x)=w(|x|)$ is a nonincreasing function of $|x|$, then $w(x)$ is regular. As a by-product, the following theorem is obtained. Let $f(t)=\sum a_ne^{int}$ $(a_0=0)$ be an absolutely convergent Fourier series such that $|a_{\pm n}|\leqq a_n^*$ $(n\geqq 1)$, where $a_n^*$ is a nonincreasing sequence of numbers with a finite sum. Then if $g(t)\sim\sum b_ne^{int}$ $(b_0=0)$ is a contraction of $f(t)$, the Fourier series of $g(t)$ also converges absolutely and, setting $a_{-n}^*=a_n^*$, $a_0^*=0$, we have $\sum|b_n|<5\sum a_n^*$. *B. de Sz. Nagy.*

**Kozlov, V. Ya. On the completeness of systems of functions $\{\phi(nx)\}$ in the space $L_2[0,2\pi]$.** Doklady Akad. Nauk SSSR (N.S.) 61, 977–980 (1948). (Russian)

We write $\varphi(x)$ for a function of period $2\pi$ and of class $L^2(0,2\pi)$ and $A(\varphi_1,\varphi_2,\cdots)$ for the set of functions $\{\varphi_1(kx)\varphi_2(kx)\cdots\}$ $(k=1,2,\cdots)$. The author studies the completeness of $A(\varphi_1,\varphi_2,\cdots)$ in $L^2(0,2\pi)$. (1) The set $A(\varphi_1,\varphi_2,\cdots)$ is a complete, normed orthogonal system if and only if $A(\varphi_1,\varphi_2,\cdots)=A(\varphi_1,\varphi_2,\varphi_3)$, $\varphi_1=1$, $\varphi_2=a\cos x+b\sin x$, $\varphi_3=c\cos x+d\sin x$, $ab+cd=0$. (2) If and only if $A(1,\varphi_1,\varphi_2,\cdots)$ is incomplete, then there exists an incomplete, normed, orthogonal set of functions $\{\mu_k(x),\nu_k(x)\}$ such that (i) $\mu_k(x)$ and $\nu_k(x)$ have Fourier expansions containing no terms in $\cos lx$ or $\sin lx$ $(l<k)$, and $\nu_k(x)$ contains no term in $\cos kx$. (ii) Every function of $A(\mu_k,\nu_k)$ can be expanded in a series of $\mu_k(x)$, $\nu_k(x)$. (iii) Every $\varphi_k(x)$ can be expanded in a similar series. If $n=\prod_\nu p_\nu^{\alpha_\nu}$ is the factorization of $n$ into primes, we define the auxiliary function $\Phi(\lambda,s)$ associated with

$\varphi=\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx)$ by

$$\Phi(\lambda, z)=\lambda_1 a_1+\lambda_1 b_1+\sum_{n=2}^{\infty}(\lambda_1 a_n+\lambda_2 b_n)\prod_{\nu} z_\nu^{\alpha_{n\nu}}=\lambda_1 X+\lambda_2 \Psi,$$

where $\lambda_1$, $\lambda_2$ are complex numbers and the variables $z$ are restricted to the region $G$: $|z_\nu|<1$, $\sum_{\nu=1}^{\infty}|z_\nu|^2<\infty$. (3) A necessary condition for the completeness of $A(1, \varphi_1, \varphi_2)$ is $\chi_1\Psi_2-\Psi_1\chi_2\neq 0$ in $G$. If $\varphi_1$ and $\varphi_2$ are trigonometric polynomials, this condition is also necessary. (4) The equation $\sum_{n=1}^{N}a_n u(nx)=f(x)$ has a solution $u(x)$ for every $f(x)\in L_2(0, 2\pi)$, if and only if $\sum_{n=1}^{N}a_n\prod_\nu z_\nu^{\alpha_{n\nu}}\neq 0$ for $|z_\nu|<1$ ($\nu=1, 2, \cdots, N$). *W. H. J. Fuchs* (Ithaca, N. Y.).

**Kozlov, V. Ya. On the completeness of systems of functions $\{\phi(nx)\}$ in the space of odd functions of $L_2[0, 2\pi]$.** Doklady Akad. Nauk SSSR (N.S.) 62, 13–16 (1948). (Russian)

The paper is a continuation of that reviewed above. The notations are the same as in the preceding review. The paper treats the problem of the completeness of $A(\varphi(x)=\sin x+\sum_2^{\infty}b_n \sin nx)$ in the space of odd functions of $L_2(0, 2\pi)$. (A slightly more general Hilbert space formulation of the problem is given.) (1) If the auxiliary function $\Phi$ satisfies (*) $|\Phi(z)-1|\leqq 1$ for $(z_\nu)\in G$, then $A(\varphi)$ is complete. If the Fourier coefficients $b_n=0$ for all $n$ except those of the form $n=kp$ ($p$ a prime), then (*) is also necessary. (2) If $0<\delta<|\Phi(z)|<M$ for all $(z_\nu)\in G$, then $A(\varphi(x))$ is a basis of $L_2(0, 2\pi)$. The remaining theorems of the paper concern functions with multiplicative or completely multiplicative coefficients. These results have for the most part been found independently and at an earlier date [Hartman, Duke Math. J. **14**, 755–767 (1947); these Rev. **9**, 426; and references given there]. If (**) $\varphi(x)=\sin x+\sum_{n=1}^{\infty}c_n \sin p^n x$ then the auxiliary function $\Phi$ is a function of one complex variable only. (3) If $\varphi$ is of the form (**) then $A(\varphi)$ is complete if and only if $\log\Phi(z)$ is the Poisson-Jensen integral of $\log|\Phi(e^{i\theta})|$. *W. H. J. Fuchs* (Ithaca, N. Y.).

**Doss, Raouf. Un théorème ergodique.** Bull. Sci. Math. (2) **72**, 76–79 (1948).

If $f(x)$ is periodic and $\xi$ arbitrary fixed, then, for $f(x)\in L$, the sequence $f_n(x)=n^{-1}\sum_{k=0}^{n-1}f(x+k\xi)$ is convergent in $L$-norm. The author extends this from periodic to almost periodic functions in $(-\infty, \infty)$, and he concentrates exclusively on Stepanoff functions with Stepanoff norm, although, as it cursorily appears to the reviewer, it could also be done for other classes (Weyl, Besicovitch, etc.), and probably for the analogues to $L_p$, for all $p\geqq 1$. The author employs certain estimates, previously established by him, for the reviewer's approximating sums. *S. Bochner.*

## *Polynomials, Polynomial Approximations*

**Erdös, P., and Turán, P. On a problem in the theory of uniform distribution. I.** Nederl. Akad. Wetensch., Proc. **51**, 1146–1154 = Indagationes Math. **10**, 370–378 (1948).

**Erdös, P., and Turán, P. On a problem in the theory of uniform distribution. II.** Nederl. Akad. Wetensch., Proc. **51**, 1262–1269 = Indagationes Math. **10**, 406–413 (1948).

In a forthcoming paper the authors prove the following theorem. If $f(z)=a_0+\cdots+a_nz^n$ satisfies $|f(z)|\leqq M$ on $|z|=1$, and has roots $z_1, \cdots, z_n$, then the number of roots with $\alpha\leqq\arg z_\nu\leqq\beta$ differs from $n(\beta-\alpha)/2\pi$ by less than $16\{n\log(M|a_0a_n|^{-\frac12})\}^{\frac12}$. It is pointed out that a similar result cannot hold in terms of $M(\theta)$, an upper bound for $|f(z)|$ on $|z|=\theta$, where $\theta$ is fixed and $0<\theta<1$. The main theorem of the present paper is that such a theorem does hold if it is further postulated that all the roots are outside $|z|=1$. If $M(\theta)=|a_0a_n|^{\frac12}\exp(n/g(n, \theta))$, where $n\geqq g(n, \theta)\geqq 2$, it is proved that the number of roots with $\alpha\leqq\arg z_\nu\leqq\beta$ differs from $n(\beta-\alpha)/2\pi$ by less than $C(\log 4\theta^{-1})(n/\log g(n, \theta))$, where $C$ is a numerical constant. The proof uses first a method of Schur to reduce the general case to that when all the roots are on $|z|=1$, and the proof of that case is based on the following theorem in Diophantine approximation. Let $\varphi_1, \cdots, \varphi_n$ be real, and suppose $|\sum_{\nu=1}^{n}e^{ik\varphi_\nu}|\leqq\psi(k)$ for $k=1, \cdots, m$. Then the number of the $\varphi_\nu$ with $\alpha\leqq\varphi_\nu\leqq\beta$ (mod $2\pi$) differs from $n(\beta-\alpha)/2\pi$ by less than $C(n/(m+1)+\sum_{k=1}^{m}\psi(k)/k)$, with a numerical constant $C$. This is a finite analogue of Weyl's criterion for equal distribution. The paper contains a number of remarks, in addition to the main theorem.

*H. Davenport* (London).

**de Bruijn, N. G. Some theorems on the roots of polynomials.** Nieuw Arch. Wiskunde (2) **23**, 66–68 (1949).

As a generalization of a result due to L. Weisner [Amer. J. Math. **64**, 55–60 (1942); these Rev. **3**, 235], the author proves that, if all the zeros of $A(z)=\sum_0^m a_kz^k$, $a_m\neq 0$, lie in the sector $\alpha_1<\arg z<\alpha_2$ with $0\leqq\alpha_2-\alpha_1\leqq\pi$ and all the zeros of $B(z)=\sum_0^n b_kz^k$, $b_n\neq 0$, lie in the sector $\beta_1<\arg z<\beta_2$ with $0\leqq\beta_2-\beta_1\leqq\pi$, then all the zeros of $C(z)=\sum_0^p k!a_kb_kz^k$ with $p=\min(m, n)$ lie in the sector $\alpha_1+\beta_1<\arg(-z)<\alpha_2+\beta_2$. This leads to the conclusion that, if $F(z)\not\equiv 0$ and $G(z)\not\equiv 0$ are polynomials each having all its zeros in the strip $S$: $|\Im z|\leqq a$, then all the zeros of $H(z)=\sum_{n=0}^{\infty}(t^n/n!)F^{(n)}(z)G^{(n)}(z)$, where $t<0$, lie in $S$. *M. Marden* (Milwaukee, Wis.).

**de Bruijn, N. G. An analogue of Grace's apolarity theorem.** Nieuw Arch. Wiskunde (2) **23**, 69–76 (1949).

The author defines a function $F(z)$ of type $R$ to be an entire function which is the limit of a sequence of polynomials possessing only real zeros. By a theorem due to Pólya and Schur [J. Reine Angew. Math. **144**, 89–113 (1914)], $F(z)$ has the form $E(z)\exp(-az^2/2)$, where $E(z)$ is an entire function of genus zero or one and $a\geqq 0$. The present note contains a statement and proof of the following two results. For any number $b>c=a/(1+a)$ but for no number $b<c$, there exists a unique function $f(y)$, continuous and $O(\exp(by^2/2))$, such that

$$(1)\qquad F(z)=(2\pi)^{-\frac12}\exp(z^2/2)\int_{-\infty}^{\infty}f(y)\exp(izy-y^2/2)dy.$$

Furthermore, if $F_1(z)$ and $F_2(z)$ are any two functions of type $R$ such that $b_1+b_2<1$ for the corresponding functions $f_1(z)$ and $f_2(z)$, then the function $F(z)$, defined by (1) with $f(z)=f_1(z)f_2(z)$, is also of type $R$. The latter theorem is first proved for polynomials $F(z)$ with the aid of the results mentioned in the preceding review and is then extended to entire functions by a limiting process. It is a composition theorem analogous to the corollary of Grace's apolarity theorem for polynomials all of whose zeros lie on a given circle or straight line. *M. Marden* (Milwaukee, Wis.).

**Popoviciu, Tiberiu. Sur certaines inégalités entre les zéros, supposés tous réels, d'un polynome et ceux de sa dérivée.** Ann. Sci. Univ. Jassy. Sect. I. **30** (1944–1947), 191–218 (1948).

Let $f(x)$ be a real polynomial with only real zeros. Let the zeros $x_i$ of $f(x)$ and the zeros $y_j$ of its derivative $f'(x)$ be arranged so as to form nondecreasing sequences. Then

$$(n-i+1)^{-1}\{(n-i)\sum_{k=0}^{j-1}x_k+jx_{i+j}\}$$

$$\leqq\sum_{k=0}^{j-1}y_{i+k}\leqq(i+j)^{-1}\{jx_i+(i+j-1)\sum_{k=1}^{j}x_k\},$$

with the equality signs valid if and only if

$$x_i=x_{i+1}=\cdots=x_{i+j}=y_i=y_{i+1}=\cdots=y_{i+j-1}.$$

This inequality, a generalization of one due to J. Nagy [Jber. Deutsch. Math. Verein. **27**, 37–43 (1918)], is proved here by the use of a lemma that the $y_j$ are nondecreasing continuous functions of the $x_i$. A corollary which follows by setting $i=1$ (or $i=n-1$) is that the average of the first (last) $j$ of the $y_i$ is not greater (less) than the average of the first (last) $j$ of the $x_i$. An additional result concerns the theorem of K. Toda [J. Sci. Hiroshima Univ. Ser. A. **4**, 27–40 (1934)] that for any function $F(x)$, nonconcave for $x_1\leqq x\leqq x_n$, $(n-1)^{-1}\sum_{k=1}^{n-1}F(y_k)\leqq n^{-1}\sum_{k=1}^{n}F(x_k)$. The same theorem is shown to hold when $n$ is replaced by $n-1$ and the $y_k$ and $x_k$ by their averages. *M. Marden.*

**Varopoulos, Th. On a theorem of Walsh.** Bull. Soc. Math. Grèce **23**, 1–2 (1948). (Greek)

An inductive proof is here given for the following theorem due to J. L. Walsh [Ann. of Math. (2) **25**, 285–286 (1924)]. All the zeros of the polynomial $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$ lie in the circle $|z|\leqq|a_1|+|a_2|^{\frac{1}{2}}+\cdots+|a_n|^{1/n}$. The proof is similar to the original one of Walsh.

*J. Dugundji and M. Marden.*

**Vuthoulkas, Dion. On a theorem of Walsh.** Bull. Soc. Math. Grèce **23**, 15–17 (1948). (Greek)

The theorem of Walsh stated in the preceding review is refined to read as follows. All the zeros of the polynomial $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$ lie in the circle

$$|z+(a_1/2)|\leqq|a_1/2|+|a_2|^{\frac{1}{2}}+\cdots+|a_n|^{1/n}.$$

This refinement is also due to Walsh [loc. cit.].

*J. Dugundji and M. Marden.*

**Vuthoulkas, Dion Pan. On the modulus of roots of polynomials.** Bull. Soc. Math. Grèce **23**, 2–14 (1948). (Greek)

(1) If the polynomial $\alpha_0+\alpha_1z^{\nu_1}+\alpha_2z^{\nu_2}+\alpha_3z^{\nu_3}$, $1\leqq\nu_1<\nu_2<\nu_3$, has one root satisfying $|\alpha_0+\alpha_1z^{\nu_1}+\alpha_2z^{\nu_2}|>|\alpha_0|$ then it has a root that satisfies $|\alpha_0+\alpha_1z^{\nu_1}+\alpha_2z^{\nu_2}|<|\alpha_0|$. (2) Given the polynomial $\varphi=\alpha_0+\alpha_1z+\cdots+z^m$, let

$$M=\max\{1, |\alpha_0/\alpha_1|, |\alpha_1/\alpha_2|, \cdots, |\alpha_{p-2}/\alpha_{p-1}|, |\alpha_{p-1}|\}.$$

Then $\varphi$ has at least $p$ roots in $|z|\leqq 2M$. The first result follows from noticing that

$$\prod_{i=1}^{\nu_3}\left|1+\frac{\alpha_1}{\alpha_0}z_i^{\nu_1}+\frac{\alpha_2}{\alpha_0}z_i^{\nu_2}\right|=\left|\frac{\alpha_3}{\alpha_0}\right|^{\nu_3}\prod_{i=1}^{\nu_3}|z_i|^{\nu_3}=1,$$

where $z_i$ are the roots, and the second from a result of T. Varopoulos. [Reviewers' note: a better bound for (2) was given by Montel [Comment. Math. Helv. **7**, 178–200 (1935)], namely, that at least $p$ zeros lie in $|z|\leqq 2N$, where

$$N=\max\{|\alpha_0/\alpha_1|, |\alpha_1/\alpha_2|, \cdots, |\alpha_{p-2}/\alpha_{p-1}|, |\alpha_{p-1}|^{1/q}\},$$

and $q=n-p+1$.] *J. Dugundji and M. Marden.*

**Throumoulopoulos, Lazaros. On the modulus of the roots of polynomials.** Bull. Soc. Math. Grèce **23**, 18–20 (1948). (Greek)

If $z_p$ is the root of $z^n+a_1z^{n-1}+\cdots+a_n$ with the largest modulus, then

$$|z_p|\geqq n^{-1}\sum_{k=1}^{n}\{|a_k|/\tbinom{n}{k}\}^{1/k}.$$

This follows from the expression of the coefficients in terms of the roots, $\sum|z_i|\geqq|a_1|$, $\sum|z_iz_j|\geqq|a_2|$, etc., so that $n|z_p|\geqq|a_1|$, $\binom{n}{2}|z_p|^2\geqq|a_2|$, etc., and then adding these latter inequalities. *J. Dugundji* (Los Angeles, Calif.).

**Georgiev, G. Sur les suites de polynomes de Sturm.** C. R. Acad. Bulgare Sci. Math. Nat. **1**, no. 1, 21–24 (1948).

The author shows that a sequence of polynominals satisfying the recurrence relation for Chebyshev polynomials are necessarily the Chebyshev polynomials associated with a nondecreasing weight function. [The result was previously given by Favard, C. R. Acad. Sci. Paris **200**, 2052–2053 (1935).] *R. P. Boas, Jr.* (Providence, R. I.).

**Nassif, M. On the mode of increase of the product of basic sets of polynomials.** Amer. J. Math. **71**, 40–49 (1949).

Associated with a basic set of polynomials are its order $\omega$ and type $\gamma$. [Much of the terminology of the present paper is found in J. M. Whittaker, Interpolatory Function Theory, Cambridge University Press, 1935.] If $\{p_n(z)\}$ and $\{q_n(z)\}$ are two basic sets of given order and type, bounds are obtained for the order and type of the product set $\{u_n(z)\}$ (if $p_n(z)=\sum_i p_{ni}z^i$, $q_n(z)=\sum_i q_{ni}z^i$, then $u_n(z)=\sum_i p_{ni}q_i(z)$), for various hypotheses on $\{p_n(z)\}$ and $\{q_n(z)\}$; and these bounds are shown by examples to be best possible. For example (to take a result where the terminology requires no lengthy explanation), if $\{p_n\}$, $\{q_n\}$ are simple sets of order $\omega_1$, $\omega_2$ and of finite type, then $\{u_n\}$ is of increase not exceeding order $\omega_1+2\omega_2$ and type zero. *I. M. Sheffer.*

## *Special Functions*

**Shah, S. M., and Sharma, U. C. Some properties of a function of Ramanujan.** J. Univ. Bombay (N.S.) **17**, part 3, sect. A, 1–4 (1948).

Let $we^{-w}=te^{-t}$, where $0\leqq w\leqq 1$, $t\geqq 1$, and define $\phi(t)=w/t$ for $t\geqq 1$. It is proved that $(-1)^k\phi^{(k)}(t)\geqq 0$, $t\geqq 1$, $k=1, 2, 3, 4$.

*H. Pollard* (Ithaca, N. Y.).

**Mayrhofer, Karl. Über die Ableitungen der Legendreschen Kugelfunktionen 2. Art in der Nähe der singulären Stellen.** Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **156**, 567–572 (1948).

The $k$th derivative of $Q_n(x)$ near $x=1$ is approximately $\frac{1}{2}(k-1)!(1-x)^{-k}$, and for $k=2$ this seems to be a good approximation, and is certainly an upper bound, for $Q_n''(x)$ in the interval $1.001\leqq x\leqq 1.1$. *A. Erdélyi.*

**Oniga, Théodore. Sur une généralisation des fonctions circulaires et hyperboliques.** C. R. Acad. Sci. Paris **227**, 1138–1140 (1948).

The functions in question are suitably selected solutions of the differential equations $d^n\Phi_{nk}/dx^n = \Phi_{nk}$, $d^n\Psi_{nk}/dx^n = -\Psi_{nk}$; their operational representations are respectively $p^{k-1}/(p^n-1)$ and $-p^{k-1}/(p^n+1)$. Expansions in exponential functions, power series expansions, addition theorems, etc., readily follow, as does the connection between $\Phi$ and $\Psi$.

*A. Erdélyi* (Pasadena, Calif.).

**Gupta, H. C. On operational calculus.** Proc. Nat. Inst. Sci. India **14**, 131–156 (1948).

In this paper a large number of integrals with Bessel functions, confluent hypergeometric functions, and other functions of the hypergeometric type are evaluated by means of four theorems which, under appropriate conditions which there is no space available to reproduce here, state the following results. (i) If $f(p) \doteqdot h(x)$ then

$$p^{\mu-\lambda}f(p^{-\mu}) \doteqdot x^{\lambda}\int_0^\infty h(s)J_\lambda^\mu(sx^\mu)ds.$$

(ii) If $f(p) \doteqdot \phi(x)$ then

$$x^{-1+\lambda/\mu}\phi(x^{-1/\mu}) \doteqdot \mu p\int_0^\infty t^{\lambda-1}J_\lambda^\mu(pt^\mu)f(t)dt.$$

(iii) If $f(p) \doteqdot h(x)$, $p^{\beta-\alpha}h(p^\beta) \doteqdot \phi(x)$ and $p^{\beta-\alpha}f(p^{-\beta}) = \psi(x^\beta)$, then $x^{-1+\alpha/\beta}\phi(x^{-1/\beta}) \doteqdot p^{1-\alpha/\beta}\psi(p)$. (iv) If $f(p) \doteqdot h(x)$ and $p^{1-\lambda+\mu}h(p^\mu) \doteqdot \phi(x)$, then

$$p^{-1}f(p) = \int_0^\infty x^{-\lambda}G_\lambda^\mu(px^{-\mu})\phi(x)dx.$$

Here $G_\lambda^\mu(x) = \mu\sum_{r=0}^\infty(-x)^r\Gamma(\lambda+\mu r)/r!$ if $\mu<1$, with similar definitions when $\mu \geqq 1$, and

$$J_\lambda^\mu(x) = \sum_{r=0}^\infty \frac{(-x)^r}{r!\Gamma(1+\lambda+\mu r)}.$$

There are 23 "results," i.e., applications of the theorems to integrals which contain a number of parameters, and numerous particular cases of these results. *A. Erdélyi.*

**Rutgers, J. G. Extension of some identities. II.** Nederl. Akad. Wetensch., Proc. **51**, 996–1004 = Indagationes Math. **10**, 329–337 (1948). (Dutch)

The first part appeared in the same vol., 868–873 = Indagationes Math. **10**, 296–301 (1948); these Rev. **10**, 296. In the present second part the case of $(-1)^\nu$, arbitrary $\nu$, and positive integer $k$ (even or odd), is dealt with.

*A. Erdélyi* (Pasadena, Calif.).

**Wood, W. L. Note on a new form of the solution of Reynolds' equation for Michell rectangular and sector-shaped pads.** Philos. Mag. (7) **40**, 220–226 (1949).

Michell [Z. Math. Phys. **52**, 123–137 (1905)] solved Reynolds' equation for finite bearing pads. The solution is given in a series which is often unwieldy. This note expresses the result in a combination of Bessel functions with Struve functions for imaginary argument. These Struve functions are tabulated. *A. E. Heins* (Pittsburgh, Pa.).

**Humbert, Pierre. Les fonctions de Mathieu et le calcul symbolique.** Bull. Sci. Math. (2) **72**, 23–32 (1948).

It has been known for some time that the Laplace transform of Mathieu's differential equation (in a suitable "algebraic" form) is essentially again Mathieu's equation. This can be confirmed, as the author shows, by the rules of operational calculus. The Laplace transform, then, of a Mathieu function is essentially another Mathieu function, but so far no detailed and precise results have been available in the general case (as distinct from the special case of periodic solutions). The author obtains such results by utilising certain integral formulae for Mathieu functions which he takes from the recent book by N. W. McLachlan [Theory and Application of Mathieu Functions, Oxford, 1947; these Rev. **9**, 31] whose notations he also follows. He finds

$$\mathrm{ce}_{2n}(\pi/2, q)t^{-\frac{1}{2}}(t+2)^{-\frac{1}{2}}\,\mathrm{Ce}_{2n}(\cosh^{-1}(1+t), q)$$
$$\supset(-)^n\pi A_0^{(2n)}pe^p\,\mathrm{Fe}\,k_{2n}(\sinh^{-1}\tfrac{1}{2}p/k, -q)$$

and similar formulae for $\mathrm{Ce}_{2n+1}$, $\mathrm{Se}_{2n}$, $\mathrm{Se}_{2n+1}$. By using well-known processes of the operational calculus he derives from these operational forms further integral formulae which are, however, too complicated for inclusion in this review.

*A. Erdélyi* (Pasadena, Calif.).

## *Harmonic Functions, Potential Theory*

**Walsh, J. L., Sewell, W. E., and Elliott, H. M. On the degree of convergence of harmonic polynomials to harmonic functions.** Proc. Nat. Acad. Sci. U. S. A. **35**, 59–62 (1949).

Several theorems are stated without proof concerning the degree of convergence of sequences of harmonic polynomials to a function harmonic within one or more mutually exterior analytic Jordan curves. Refinements of existing results are obtained by studying the effect, upon the degree of convergence, of continuity properties of the harmonic function on the boundary of the point set under consideration.

*E. N. Nilson* (Hartford, Conn.).

**Fichera, G. Sul flusso di una funzione armonica.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 402–407 (1948).

Dans l'espace à $r$ dimensions, on considère un domaine borné $D$ limité par deux surfaces fermées $\Sigma_1$, $\Sigma_2$ ($\Sigma_2$ enveloppant $\Sigma_1$). Étendant le résultat analogue de Picone dans le plan, l'auteur montre que si une fonction harmonique $u$ prend sur $\Sigma_1$ et $\Sigma_2$ des valeurs, appartenant à des intervalles respectifs séparés, le flux de $u$ à travers une surface $\Sigma$ tracée dans $D$ et contenant $\Sigma_1$ est différent de 0. Se ramenant aussitôt à des surfaces assez régulières, il montre que la condition de nullité du flux équivaut à ce que $u$ soit un potentiel de double couche classique sur les surfaces, en utilisant un système d'équations intégrales. Mais signalons que la proposition annoncée à laquelle il passe ensuite, se ramène aussitôt au cas de $u>0$ s'annulant sur l'une des surfaces et l'on sait que si cette surface est assez régulière, il y a une dérivée normale positive. On peut aussi sans hypothèse de régularité des surfaces ni même de continuité de $u$ à la frontière (où l'on considèrera seulement les limites supérieure et inférieure) raisonner comme suit: si $K$ est intermédiaire aux intervalles, l'enveloppe supérieure $v$ de $K$ et $u$ est sousharmonique, de masses associées situées sur le lieu $u=K$, à distance positive de $\Sigma_1$ et $\Sigma_2$; la masse totale non nulle vaut la différence des flux de $v$ (à un facteur près) à travers deux surfaces respectivement voisines de $\Sigma_1$ et $\Sigma_2$; l'un de ces flux est nul à cause de la constance de $v$ au voisinage de l'une des surfaces; l'autre est le flux de l'énoncé. *M. Brelot* (Grenoble).

**Lorentz, G. G. Über die Dichte des Potentials einer räumlichen Belegung.** Math. Z. **51**, 262–264 (1948).

If the potential of a mass distribution in a three-dimensional space is known, the density of the distribution is determined by Poisson's equation: $\nabla^2 u = -4\pi\rho$. The author discusses an extension of this relation in which $\nabla^2 u$ is replaced by a generalized Laplacian. He notes that his results may be extended to $n$-dimensional space.

*F. W. Perkins* (Hanover, N. H.).

**Cattaneo, C. Su alcuni potenziali di strato e su qualche loro applicazione.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 416–420 (1948).

L'auteur étudie le potentiel dans l'espace d'une couche simple sur un disque plan de rayon $R$, de densité superficielle $(R^2-r^2)^{\frac{s}{2}}$ à la distance $r$ du centre $(s=-1, 1, 3, 5, 7, \cdots)$. Il en déduit une solution explicitée à l'aide de ces fonctions du problème de la détermination d'une fonction harmonique dans un demi-espace, s'annulant à l'infini, se réduisant à un polynome quelconque $P$ sur un disque $D$ du plan-frontière avec dérivée normale nulle sur le reste $D_1$ du plan, ou bien se réduisant à 0 sur $D_1$ avec dérivée normale $P$ sur $D$. Il en fait une application à certains petits mouvements d'un fluide incompressible. Les démonstrations seront publiées ultérieurement.

*M. Brelot* (Grenoble).

**Hondl, Stanko. The attraction of a uniform spherical surface.** Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fis. Astr. Ser. II. **3**, 97–113 (1948). (Croatian. English summary)

**Potts, D. H. Generalized Laplacians of higher order.** Duke Math. J. **15**, 947–951 (1948).

The author defines generalized Laplacians:

$$\nabla_p^{(n)}f(P) \equiv \lim_{r\to 0} 4r^{-2}[L(\nabla_p^{(n-1)}f; P, r) - \nabla_p^{(n-1)}f(P)],$$

$$\nabla_a^{(n)}f(P) \equiv \lim_{r\to 0} 8r^{-2}[A(\nabla_a^{(n-1)}f; P, r) - \nabla_a^{(n-1)}f(P)],$$

where $\nabla_p^{(0)}f(P) = \nabla_a^{(0)}f(P) \equiv f(P)$ and $L(f; P, r)$ and $A(f; P, r)$ are the mean values of $f(P) \equiv f(x, y)$ on the perimeter and interior, respectively, of a circle with center $P$ and radius $r$. These generalize operators due to Privaloff and Blaschke [see Potts, Bull. Amer. Math. Soc. **54**, 782–787 (1948); these Rev. **10**, 39]. Extensions of known results for smooth functions are obtained; a typical result is the following one. If $f(P)$ is continuous in a bounded domain $D$, if $\lim_{r\to 0} A(\nabla_p f; P, r)$ everywhere on $D$, if $|\nabla_p f(P)|^2$ is superficially summable on $D$, and if $\nabla_p^{(2)}f(P)$ exists at a point $R$ in $D$, then $L(f; P, r) = f(R) + \frac{1}{4}r^2\nabla_p f(R) + \frac{1}{64}r^4\nabla_p^{(2)}f(R) + o(r^4)$. The author's results are comparable to expansions found by Nicolesco [Les Fonctions Polyharmoniques, Actual. Sci. Ind., no. 331, Hermann, Paris, 1936, pp. 7–8].

*M. O. Reade* (Ann Arbor, Mich.).

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## *Differential Equations*

**✶Tricomi, Francesco. Equazioni Differenziali.** Giulio Einaudi, Torino, 1948. 313 pp.

This is a very well planned and written textbook in ordinary differential equation theory for graduate students. The five chapters into which it is divided are devoted, respectively, to: (1) the fundamental theorems concerning the existence, uniqueness, and continuity properties of solutions; (2) the qualitative properties of the family of characteristics for an equation of the first order; (3) boundary value problems for linear equations of the second order; (4) asymptotic properties of eigenfunctions and eigenvalues for linear equations of the second order; (5) differential equations in the complex domain. The author does not aim at any high degree of completeness, and it is only rarely that he assumes any knowledge on the part of the reader that could not be obtained from an ordinary introductory course in the theory of functions. Within these limits, however, the treatment is sound and remarkably clear. The lucid exposition in the second and third chapters appealed strongly to the reviewer. There are no problems for the reader, but there are numerous illuminating applications of the general theory to some of the particular equations which are important in mathematical physics. There seem to be almost no typographical errors; and the physical appearance of the volume is excellent.

*L. A. MacColl.*

**Cafiero, Federico. Su di un teorema di Montel relativo alla continuità, rispetto al punto iniziale, dell'integrale superiore ed inferiore di una equazione differenziale.** Rend. Sem. Mat. Univ. Padova **17**, 186–200 (1948).

The author shows that any group of conditions that are sufficient to assure the existence of a superior and inferior integral of the differential equation $y' = f(x, y)$ is also sufficient to assure the continuous dependence of the superior (inferior) integral on the initial value lying in the superior (inferior) region. The theorem of Montel referred to appears in Bull. Sci. Math. (2) **50**, 205–217 (1926), where further references are given.

*R. Bellman.*

**Trevisan, Giorgio. Un teorema per i sistemi di due equazioni differenziali ordinarie.** Rend. Sem. Mat. Univ. Padova **17**, 219–221 (1948).

Consider the differential equation

$$y' = f(x, y, z), \quad z' = g(x, y, z),$$

where $f$ and $g$ satisfy the Lipschitz conditions

$$|f(x, y_1, z) - f(x, y_2, z)| \leqq h(x)|y_1 - y_2|,$$
$$|g(x, y, z_1) - g(x, y, z_2)| \leqq h(x)|z_1 - z_2|,$$

with $h(x)$ integrable over $(a, b)$, and are nondecreasing in $z$ and $y$, respectively; $[f(x, y, z_1) - f(x, y, z_2)][z_1 - z_2] \geqq 0$, $[g(x, y_1, z) - g(x, y_2, z)][y_1 - y_2] \geqq 0$. Under these conditions, it is demonstrated that if $\phi(x) = [y_1(x) - y_2(x)][z_1(x) - z_2(x)]$ satisfies the boundary conditions $\phi(a) = \phi(b) = 0$, then $\phi(x) \equiv 0$ in the interval $a \leqq x \leqq b$, where $y_1$, $z_1$, and $y_2$, $z_2$ are two solutions of the above equation.

*R. Bellman.*

**Michal, Aristotle D. Solutions of systems of linear differential equations as entire analytic functionals of the coefficient functions.** Math. Mag. **22**, 57–66 (1948).

A first-order linear differential system in $n$ unknown functions $w_i(x)$ is written in the matrix form $dw/dx = A(x)w(x)$, $w(a) = w_0$, $A(x)$ denoting an $n \times n$ matrix of continuous functions. The solution is expressed in the form

$$w[A(s)|x] = \Omega_a^x[A(s)]w_0,$$

where $\Omega_a^x[A(s)]$ is the matrizant functional of the matrix $A(s)$. The paper is concerned with the Fréchet differentiability and analyticity of the matrizant and of the solution of the differential system as functions of the matrix $A(s)$. Total Fréchet differential equations characterizing these functionals are given.

*A. E. Taylor* (Los Angeles, Calif.).

**Wallach, Sylvan. On the location of spectra of differential equations.** Amer. J. Math. **70**, 833–841 (1948).

The author considers the boundary problem

$$(*)\qquad x''+(\lambda+f(t))x=0,\quad x(0)\sin\alpha+x'(0)\cos\alpha=0,$$

where $f(t)$ is real and continuous for $0\leqq t<\infty$. If

$$\int^{\infty} tf^2(t)dt<\infty,$$

then (1) (*) is in the limit point case, (2) no positive $\lambda$ is in the point spectrum, (3) every $\lambda\geqq 0$ is in the continuous spectrum, (4) no $\lambda<0$ is in the continuous spectrum, (5) at most $\lambda=0$ is in the cluster spectrum, (6) the point spectrum is a bounded set for a fixed $\alpha$. If as $t\to\infty$, $\limsup t|f(t)|=b<\infty$ then (1), (4) and (6) are true, and no $\lambda>b^2$ is in the point spectrum while every $\lambda\geqq b^2$ is in the continuous spectrum. It is shown that in a sense (2) is a best possible result. *N. Levinson* (Cambridge, Mass.).

**Wallach, Sylvan. The spectra of periodic potentials.** Amer. J. Math. **70**, 842–848 (1948).

The differential equation (*) $x''+(\lambda+f(t))x=0$ is considered, where $f(t)$ is a real continuous function of period 1. The characteristic roots of (*) are $\rho(\lambda)$ and $1/\rho(\lambda)$. The values of $\lambda$ associated with $\rho(\lambda)=\pm1$ divide the real $\lambda$-axis into two sets of open intervals. In one of these sets are values of $\lambda$ for which (*) has stable solutions and in the other, unstable solutions. Consider now the boundary conditions $x(0)=x(n)=0$, where $n$ is an integer. The author generalizes and completes a result of Wirtinger by showing that the spectrum $S^n=\{\lambda_i^n\}$, $i=0, 1, 2, \cdots$, has the properties that (1) $\lambda$ is in $S^n$ if $\rho(\lambda)$ is a complex $2n$th root of unity, (2) $\rho(\lambda_{mn-1}^n)$ is real for $m=1, 2, \cdots$, (3) $\lambda_{mn-1}^n=\lambda_{m-1}^1$ for $n=1, 2, \cdots$, and (4) $\rho(\lambda_i^n)$ is a complex $2n$th root of unity for $i\neq mn-1$, $m=1, 2, \cdots$. The case of the infinite interval is considered, where $x(0)=0$ and $x(t)\epsilon L^2(0, \infty)$. Here it is shown that the continuous spectrum consists of the closure of the set of $\lambda$ values which are stable while the point spectrum is contained in $\{\lambda_{m-1}^1\}$ and the cluster spectrum is at most $\lambda=\infty$. Other results are given.
*N. Levinson* (Cambridge, Mass.).

**Hille, Einar. Non-oscillation theorems.** Trans. Amer. Math. Soc. **64**, 234–252 (1948).

The author considers (*) $y''+f(x)y=0$. It is assumed that $f(x)$ is integrable over $(\epsilon, 1/\epsilon)$ for any $\epsilon>0$. The author proves that if $xf(x)\epsilon L(1, \infty)$ then (*) has a solution $y(x)\to1$ as $x\to\infty$. [This result, with a change of variable, is contained in work of O. Dunkel, Proc. Amer. Acad. Arts Sci. **38**, 341–370 (1902), in particular, p. 370]. If $f(x)$ has a fixed sign for large $x$ and if there is a solution $y(x)$ such that $y(x)\to1$ as $x\to\infty$ then $xf(x)\epsilon L(1, \infty)$. The equation (*) is said to be nonoscillatory over $(a, \infty)$, $a\geqq0$, if no solution changes sign more than once in $(a, \infty)$. Let $g(x)=x\int_x^\infty f(t)dt$. If the inferior and superior limits of $g(x)$ as $x\to\infty$ are $g_*$ and $g^*$, then the conditions $g_*\leqq\frac14$, $g^*\leqq1$ are necessary for (*) to be non-oscillatory for large $x$, while the condition $g^*<\frac14$ is sufficient. By examples it is shown that all inequalities are sharp. *N. Levinson* (Cambridge, Mass.).

**Hartman, Philip. On the linear logarithmico-exponential differential equation of the second order.** Amer. J. Math. **70**, 764–779 (1948).

The author characterizes completely the behavior of the solutions of $x''+f(t)x=0$ for large values of $t$ in the case in which $f(t)$ is a logarithmico-exponential function ($L$-function). Here two cases remained undecided, that in which $\lim_{t\to\infty} f(t)=\pm\infty$ and that in which $\lim_{t\to\infty} f(t)=0$ but $tf(t)$ is not in $L(a, \infty)$. The author shows that these cases can be reduced to the known ones by a repeated use of the transformation of Liouville. The main steps of the reduction follow. Let $l_1(t)=\log t$, $l_{k+1}(t)=\log l_k(t)$, $k=1, 2, \cdots$, and $L_0(t)=1$, $L_n(t)=\prod_1^n[l_k(t)]^{-2}$. Determine

$$(1)\qquad \lambda_n=\lim_{t\to\infty}\{4t^2f(t)-\sum_0^n L_k(t)\}\{L_n(t)\}^{-1}$$

which exists for arbitrary $L$-functions $f(t)$ if $\lambda_n=\pm\infty$ is admitted. It is shown that there is a least nonnegative integer $n=M(f)$ for which $\lambda_M\neq0$ while $\lambda_k=0$ for $k=0, 1, \cdots, M-1$. Further set $F_0(t)=f(t)$. If $F_0, F_1, \cdots, F_n$ have been defined set $P_{-1}(t)=1$, $P_n(t)=\prod_1^n|F_k(t)|^{\frac12}$. If $P_n(t)\epsilon L(a, \infty)$ for large $a$, the process ends, otherwise set for large $t$

$$(2)\qquad F_{n+1}(t)=\pm1+|F_n|^{-\frac14}P_{n-1}^{-1}\frac{d}{dt}\left\{P_{n-1}^{-1}\frac{d}{dt}|F_n|^{-\frac14}\right\},$$

where the sign is that of $F_n(t)$ for large $t$. Let $N=N(f)$ be the smaller of $M(f)$ and the least negative integer $n$, if any, for which $P_n(t)\epsilon L(a, \infty)$, so that $0\leqq N\leqq M$. If $P_N(t)\epsilon L(a, \infty)$ and $N=0$ there are two solutions of the differential equation of the form $1+o(1)$ and $[1+o(1)]t$ which for $N>0$ are replaced by $[1+o(1)]t^{\frac12\pm\frac12}[L_{N-1}(t)]^{\frac12}$. If $P_N(t)\epsilon L(a, \infty)$ so that $N=M$ the solutions are of the form

$$[1+o(1)]\{P_{N-1}(t)\}^{-\frac12}\exp\left\{\pm\int^t P_N(s)ds\right\}$$

provided $\lambda_N<0$. If $\lambda_N>0$ instead, a factor $i$ appears before the integral and the remainder term has to be modified. The author gives applications to the existence of oscillatory solutions (extensions of Kneser's theorem), $L_2$-solutions in the oscillatory case, etc. [Reference could be made to the reviewer's note, Proc. Nat. Acad. Sci. U. S. A. **10**, 488–493 (1924), for the use of Liouville's transformation and for the case $tf(t)\epsilon L(a, \infty)$. For the latter see also the reviewer's remarks in the review of Wintner, same J. **69**, 87–98 (1947); these Rev. **8**, 381; and his paper reviewed above. The latter also contains extensions of Kneser's theorem and a study of the role of the lim inf and lim sup of the expression in (1) for the oscillation problem.] *E. Hille.*

**Hartman, Philip. Differential equations with non-oscillatory eigenfunctions.** Duke Math. J. **15**, 697–709 (1948).

If $q(t)$, where $0\leqq t<\infty$, is a continuous function for which there exists a number $\mu$ such that the equation $y''+(\mu-q(t))y=0$ with boundary condition $\alpha y(0)+\beta y'(0)=0$ has a solution with a finite number $n$ of zeros on $0<t<\infty$ then (1) the equation $x''+(\lambda-q(t))x=0$ is in the Grenzpunktfall, (2) subject to $\alpha x(0)+\beta x'(0)=0$ there are exactly $n-1$ eigenvalues $\lambda_0<\lambda_1<\cdots$ less than $\mu$ (or none if $n=0$), (3) an eigenfunction belonging to $\lambda=\lambda_k$ has exactly $k$ zeros, and (4) there are no points of the continuous spectrum less than $\mu$. The author thereby obtains a result of Weyl with a weaker hypothesis. *N. Levinson* (Cambridge, Mass.).

**Hartman, Philip, and Putnam, Calvin R. The least cluster point of the spectrum of boundary value problems.** Amer. J. Math. **70**, 849–855 (1948).

The authors consider the differential equation

$$y''+(\lambda-q(x))y=0$$

for real continuous $q(x)$, $0\leqq x<\infty$ and real $\lambda$. The equation

is oscillatory or nonoscillatory for a fixed $\lambda$ according as a real solution has or does not have an infinite number of zeros for $x \geqq 0$. The real number $\mu$ is defined as follows: (1) if (*) is nonoscillatory for all $\lambda$ let $\mu=\infty$, (2) if (*) is nonoscillatory for some but not all $\lambda$ let $\mu$ be the unique number with the property that (*) is oscillatory when $\lambda>\mu$ and nonoscillatory when $\lambda<\mu$, (3) if (*) is oscillatory for all $\lambda$ let $\mu=-\infty$. Then in the cases (1) and (2) the cluster spectrum of (*) has $\mu$ as its least point and (*) is in the limit-point case while in the case (3) $\mu$ is the least point of the cluster spectrum if (*) is in the limit-point case. The theorem also applies to the case $(py')'+(\lambda-q)y=0$.

*N. Levinson* (Cambridge, Mass.).

**Saharnikov, N. A. On Frommer's conditions for the existence of a center.** Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 669–670 (1948). (Russian)

Frommer [Math. Ann. **109**, 395–424 (1934)] gave three sets of conditions on the coefficients of the differential equation $y'=-(x+ax^2+(2b+\alpha)xy+cy^2)/(y+bx^2+(2c+\beta)xy+dy^2)$ under which the point $x=y=0$ is a center. It was shown by Bautin [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 669–672 (1939); these Rev. **2**, 49] that some of these criteria were erroneous. The author now gives necessary and sufficient conditions under which the point $x=y=0$ is a center. These conditions are expressed in terms of algebraic relations connecting the coefficients. *R. Bellman.*

**Haag, Jules. Sur la synchronisation des systèmes à plusieurs degrés de liberté.** Ann. Sci. École Norm. Sup. (3) **64** (1947), 285–338 (1948).

This is a study, from a very general point of view, of some of the problems that are important in connection with the design of clocks and similar mechanisms. The fundamental problem can be stated as follows. A dynamical system, which admits of oscillatory motion, and which is subjected to retarding frictional forces, is kept in motion by an escapement mechanism, or by an applied force which varies periodically with the time. It is required to determine whether or not the resulting motion, starting from given initial conditions, approaches a stable periodic motion asymptotically. The author bases his discussion of this problem upon a theorem which he has published previously [Bull. Sci. Math. (2) **70**, 155–172 (1946); these Rev. **9**, 92]. For the sake of this review, it will suffice to state this theorem, somewhat incompletely, in the following terms. Consider the system of differential equations

$$(1) \qquad dx_i/dt=f_i(x,t), \qquad i=1,\cdots,n,$$

where the right-hand members are periodic functions of $t$ with the common period $T$. Also consider the associated system of equations

$$(2) \qquad dx_i/dt=F_i(x)=T^{-1}\int_0^T f_i(x,t)dt, \quad i=1,\cdots,n.$$

Let $P^0:(x_1^0,\cdots,x_n^0)$ be a point at which all of the functions $F_i(x)$ vanish. Then, in order that the system of equations (1) shall have a stable periodic solution in the neighborhood of the point $P^0$, it is necessary and sufficient that $x_i=x_i^0$ $(i=1,\cdots,n)$ be a stable solution of the system of equations (2).

After having discussed the fundamental problem stated above, and various related questions, in these general terms, the author proceeds to the consideration of a variety of specific problems relating to the design of time-keeping devices. Here he obtains a large amount of quantitative information, which is compared, as far as that is possible, with the experimental results that have been obtained by other investigators. *L. A. MacColl* (New York, N. Y.).

**Bautin, N. N. On the behavior of dynamical systems with small violations of the condition of stability of Routh-Hurwitz.** Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 613–632 (1948). (Russian)

Consider a real system $dx_i/dt=\sum a_{ij}x_j+X_i(x_1,\cdots,x_n)$, $i=1,\cdots,n$, where the $a_{ij}$ are constant and $X_i$ is a power series holomorphic around the origin and beginning with terms of degree at least two. It is known that if the characteristic roots $\lambda_i$ of $A=\|a_{ij}\|$ have their real parts negative, the origin is a stable singular point. Liapounoff in his memoir on these questions [Ann. Fac. Sci. Univ. Toulouse (2) **9**, 203–474 (1907), reprinted as Ann. of Math. Studies, no. 17, Princeton, 1947; these Rev. **9**, 34] has also treated the exceptional cases where: (a) one of the $\lambda_i$ is zero; (b) two of them are conjugate and pure complex. The author returns to (b) which he discusses rather fully for $n=2$ when the coefficients depend upon a certain parameter $\mu$. He is particularly interested in the following type of question. Suppose that for $\mu=0$ there is a bifurcation and the system passes from a stable singular point at the origin, say a stable focus, to an unstable point; is the instability thus arising dangerous or not? For instance, when in the phase plane the focus becomes unstable but there arises a very small stable limit-cycle, it is a consequence of Poincaré's general theory on these questions that the system will continue to remain not far from the origin, for $\mu$ sufficiently small, and so the new instability will not be very dangerous. The author applies some of his results to the stability of an airplane in vertical symmetrical flight with constant angle of attack. *S. Lefschetz.*

**Kalinin, S. V. On the stability of periodic motions in the case when one root is equal to zero.** Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 671–672 (1948). (Russian)

Consider a real system of the general Liapounoff type,

$$dx/dt=X, \quad dx_i/dt=\sum p_{si}x_i+X_s, \qquad s=1,2,\cdots,n,$$

where $X$, $X_i$ are holomorphic power series in $x$ and the $x_i$ in $\sum x_i^2<A$, $t>t_0$, and begin with terms of degree not less than 2 in $x$ and the $x_i$. Moreover, the coefficients of the series are bounded periodic functions of $t$ with period $\omega$. Let $c_{sr}$ be the average of $p_{sr}$ over a period and suppose that: (a) $p_{sr}=c_{sr}+\epsilon f_{sr}(t)$; (b) the characteristic roots of $\|c_{sr}\|$ all have negative real parts; (c) $X=(g+\epsilon G)x^2+P_s+Q+R$, $P=\sum(a_s+2\epsilon A_s)x_s$, $Q=\sum(b_{sr}+\epsilon B_{sr})x_rx_s$, where $g$, $a_s$ and $b_{sr}$ are constant, $G$, $A_s$, $B_{sr}$ have zero averages over a period, and $R$ begins with terms of degree at least three in $x_1,\cdots,x_n$. Using Liapounoff's so-called second method the author shows that for $\epsilon$ sufficiently small the undisturbed motion is stable when $g>0$ and unstable when $g<0$. [See Liapounoff's memoir, Ann. Fac. Sci. Univ. Toulouse (2) **9**, 203–474 (1907), reprinted as Ann. of Math. Studies, no. 17, Princeton, 1947; these Rev. **9**, 34.] *S. Lefschetz.*

**Wasow, Wolfgang. The complex asymptotic theory of a fourth order differential equation of hydrodynamics.** Ann. of Math. (2) **49**, 852–871 (1948).

The paper deals with the ordinary linear differential equation (1) $N(y)+\lambda^2M(y)=0$, in which $N(y)$ and $M(y)$ are linear differential operators of the fourth and second orders, respectively, i.e., $N(y)=\sum_{k=0}^4 a_k(x)y^{(4-k)}$,

$M(y)=\sum_{k=0}^{2} b_k(x)y^{(2-k)}$. The variable $x$ is taken to be complex, and the coefficients $a_k(x)$, $b_k(x)$ to be analytic in a neighborhood of $x=0$, with $a_0(x)\equiv 1$. Essential is the hypothesis that $b_0(0)=0$, $b_0'(0)\neq 0$. The parameter $\lambda$ is large and on a fixed ray of the complex plane. With $Q(x)=\int_0^x\{-b_0(x)\}^{\frac{1}{2}}dx$, the $x$-plane near the origin contains three curves $C_1$, $C_2$, $C_3$ which issue from $x=0$ and on which the real part of $\lambda Q(x)$ is zero. These define three curvilinear sectors $S_1$, $S_2$, $S_3$, the notation being chosen so that $C_k$ is not a boundary of $S_k$.

The facts established are roughly the following. (i) For each $k$ there exists a solution of (1) which with the determination of $\lambda Q(x)$ which has negative real part in $S_k$ has the asymptotic form $e^{\lambda Q(x)}\sum_\nu \sigma_\nu(x)\lambda^{-\nu}$ except on the curve $C_k$. (ii) If $u(x)$ is any solution of the equation $M(y)=0$, there exists for each $k$ a solution of (1) which differs from $u(x)$ by a term of the order of $\lambda^{-2}$ except in the sector $S_k$. (iii) If the function $u(x)$ referred to in (ii) is multiple-valued about $x=0$, the solution of (1) there referred to diverges at every interior point of $S_k$. (iv) If $v(x)$ is a single-valued solution of $M(y)=0$, and is unique as such, there exists a solution of (1) which differs from $v(x)$ by a term of the order of $\lambda^{-2}$ in the whole neighborhood of $x=0$.

Some boundary problems in which these asymptotic solutions find application are briefly discussed, and a role of equations of the type (1) in hydrodynamics is outlined.

*R. E. Langer* (Madison, Wis.).

**Kallmann, Hartmut, und Päsler, Max. Neue Behandlungs- und Darstellungsmethode wellenmechanischer Probleme.** Ann. Physik (6) 2, 292–304 (1948).

The authors apply the Laplace transform to find solutions of an ordinary differential equation which arises in the radial solution of the H atom. *A. E. Heins*

**Kallmann, Hartmut, und Päsler, Max. Allgemeine Behandlung des H-Atoms mit beliebigen Anfangsbedingungen mittels der Laplace-Transformation und deren physikalische Bedeutung.** Ann. Physik (6) 2, 305–320 (1948).

This paper is a continuation of the paper reviewed above, with applications similar to those found there.

*A. E. Heins* (Pittsburgh, Pa.).

**Petrovitch, Michel. Addition au mémoire sur les équations différentielles algébriques.** Acad. Serbe Sci. Publ. Inst. Math. 1, 1–4 (1947).

The paper in question appeared in Publ. Math. Univ. Belgrade 6–7, 290–325 (1938).

**Bilimovitch, Anton. Sur l'accroissement pur de la forme différentielle et son application.** Acad. Serbe Sci. Publ. Inst. Math. 1, 49–57 (1947).

For a differential form $\Phi\equiv\sum_{j=1}^{3}X_j(x_1,x_2,x_3)dx_j$ the "pure" partial derivative of $\Phi$ with respect to $x_k$ is defined to be

$$\frac{\partial\cdot\Phi}{\partial\cdot x_k}\equiv\sum_{j=1}^{3}\left(\frac{\partial X_j}{\partial x_k}-\frac{\partial X_k}{\partial x_j}\right)dx_j.$$

The author states that to find the minima of $\Phi$ one equates all the "pure" partial derivatives of $\Phi$ to zero. To justify this last statement he applies his theory to a problem in the calculus of variations. *F. G. Dressel* (Durham, N. C.).

**Zaremba, S. K. On first integrals of differential equations.** J. London Math. Soc. 23, 310–314 (1948).

Let $X$, $Y$, $Z$ be three functions of $x$, $y$, $z$ of class $C^\infty$ in a region $R$, such that $|X|+|Y|+|Z|>0$, and consider the system of differential equations (1) $X^{-1}dx=Y^{-1}dy=Z^{-1}dz$. A function $F(x,y,z)$, of class $C'$ at least, is called a first integral of (1) on a subset $\omega$ of $R$ if $X\partial F/\partial x+Y\partial F/\partial y+Z\partial F/\partial z=0$ in $\omega$. Ważewski conjectured that a necessary and sufficient condition for the existence of two functionally independent integrals of (1) of class $C^\infty$ in an open set $\omega$ of $R$ was that both half-characteristics of (1) starting from any point of $\omega$ reaches the boundary of $\omega$. Earlier the author [J. Math. Pures Appl. (9) 19, 411–426 (1940); these Rev. 3, 40] proved this condition not necessary. The purpose of the present paper is to show this condition is not sufficient.

*F. G. Dressel* (Durham, N. C.).

***Kamke, E. Differentialgleichungen. Lösungsmethoden und Lösungen. II. Partielle Differentialgleichungen erster Ordnung für eine gesuchte Funktion.** Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Band 18. Akademische Verlagsgesellschaft, Leipzig, 1944; J. W. Edwards, Ann Arbor, Mich., 1946. xv+243 pp. $6.50.

This is the second volume of an extensive reference work on differential equations. The first volume, which is devoted to ordinary differential equations, has been reviewed previously [these Rev. 9, 33]. The present volume deals in the same manner with partial differential equations of the first order in one unknown function. The first part of the book, amounting to something less than two-thirds of the whole, gives a concise presentation of the concepts, theorems, methods, etc., that are useful in the study of equations of the kind that is under consideration. This part constitutes a very good summary of the theory of these equations. The remainder of the book consists of a table, giving the solutions of approximately 300 particular equations. The excellent qualities which marked the first volume have been fully maintained. Although the second volume will probably prove to be of less general interest and usefulness than its predecessor, this must be attributed solely to the nature of the subject matter, and in no wise to any fault on the part of the author. *L. A. MacColl.*

**Nordon, Jean. Sur une solution nouvelle de l'équation de Fourier.** C. R. Acad. Sci. Paris 228, 167–168 (1949).

The author points out that solutions of the heat equation $u_{xx}=u_t$ of the type $u=t^m f(z)$, $z=x(4t)^{-\frac{1}{2}}$, can be written in terms of Weber functions. *F. G. Dressel.*

**Krasnooshkin, P. E. The interaction of oscillating systems with distributed parameters.** Vestnik Moskov. Univ. 1947, no. 1, 59–70 (1947). (Russian. English summary)

The paper appeared in English in Acad. Sci. USSR. J. Physics 9, 439–446 (1945); these Rev. 7, 303.

**Yakovlev, I. A. Hadamard's problem and the connection between equations of hyperbolic type and spaces of constant curvature.** Doklady Akad. Nauk SSSR (N.S.) 62, 175–178 (1948). (Russian)

Hadamard [Lectures on Cauchy's Problem, Yale University Press, 1923] proposed the problem of determining all linear partial differential equations of hyperbolic type for which Huyghens' principle is valid (i.e., all equations which characterize the propagation of disturbances without diffusion). Hadamard showed that Huyghens' principle does not hold for hyperbolic equations in Euclidean spaces of odd dimension, and conjectured that every "diffusion free" equation could be reduced to the wave equation by a transformation of type (A), consisting of: (1) multiplying the

given equation by a function $f(x_1, \cdots, x_n)$; (2) introducing a new dependent variable $u=\lambda(x_1, \cdots, x_n)v$; (3) introducing new independent variables. M. Mathisson [Acta Math. 71, 249–282 (1939); these Rev. 1, 120] showed that the "diffusion free" equations of the form

$$\frac{\partial^2 u}{\partial x_0^2}-\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}\right)+\sum_{i=0}^{3} B_i\frac{\partial u}{\partial x_i}+Cu=0$$

are precisely the wave equation and those equations reducible to it by a transformation of type (A), where (3) is the Lorentz transformation. The author considers Hadamard's problem for spaces of constant curvature, and asserts the correctness of Hadamard's conjecture in any three-dimensional space of constant curvature. The starting point of the argument is a result of M. N. Olevsky [C. R. (Doklady) Acad. Sci. URSS (N.S.) 45, 95–98 (1944); these Rev. 6, 230] that the mean value $M(u; p, s)$ of a twice continuously differentiable function on a quasisphere $F_s(p)$ of geodesic radius $s$ and center $p(x_1^0, x_2^0, x_3^0)$ (in the sense of the metric of a space of constant curvature $k$), satisfies the equation $\Delta^2 M-\partial^2 M/\partial s^2=(2k^{\frac12}\cot(k^{\frac12}s))\partial M/\partial s$, and the initial conditions $M|_{s=0}=u(x_1^0, x_2^0, x_3^0)$, $\partial M/\partial s|_{s=0}=0$, where $\Delta^2$ is Beltrami's second differential parameter. From this it is easily seen that the solution of Cauchy's problem for the equation (*) $\Delta^2 u-ku=\partial^2 u/\partial t^2$ with initial conditions $u|_{t=0}=u_0(x_1, x_2, x_3)$, $\partial u/\partial t|_{t=0}=u_1(x_1, x_2, x_3)$, is given by

$$u=k^{-\frac12}\sin k^{\frac12}tM(u_1; p, t)+\frac{\partial}{\partial t}\{k^{\frac12}\sin k^{\frac12}tM(u_0; p, s)\}.$$

Thus equation (*) is seen to be diffusion free. The transformations of type (A) which transform equation (*) to the wave equation are then given explicitly, the cases of positive and negative curvature $k$ being handled separately. The argument is then completed by using Mathisson's result quoted above to show that all diffusion free equations of the form

$$\Delta^2 u+\sum_{i=0}^{3} B_i\frac{\partial u}{\partial x_i}+Cu-\partial^2 u/\partial t^2=0$$

are precisely those which can be reduced to the wave equation by a transformation of type (A). *J. B. Diaz.*

**Cinquini-Cibrario, Maria. Sopra un nuovo problema ai limiti per un sistema di equazioni alle derivate parziali.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 10(79), 103–111 (1946).

The problem of determining integral surfaces of an $n$th order partial differential equation, which pass through a given characteristic strip, has been reduced by the author to solving a boundary value problem for a certain "canonical" system of equations. [See the following review.] The present note supplies the proof of the existence of solutions of that boundary value problem. *F. John.*

**Cinquini-Cibrario, Maria. Teoria delle caratteristiche per equazioni non lineari di ordine $n$ di tipo iperbolico.** Ann. Mat. Pura Appl. (4) 26, 95–117 (1947).

The author considers the differential equation

$$(1)\qquad F(x, y, z, p_{r,s})=0,\qquad 1\leqq r+s\leqq n,$$

for $z=z(x, y)$, where $p_{r,s}=\partial^{r+s}z/\partial x^r\partial y^s$, and $F$ has continuous 3d derivatives satisfying a Lipschitz condition. It is assumed that (1) is "hyperbolic," i.e., that the equation

$$(2)\qquad P_{n,0}\rho^n-P_{n-1,1}\rho^{n-1}+\cdots+(-1)^n P_{0,n}=0,$$

with $P_{r,s}=\partial F/\partial p_{r,s}$, has $n$ distinct real roots. For a given integral surface $S$ the equation $dy/dx=\rho$ with $\rho$ satisfying (2) defines a "characteristic curve" on $S$. Along a characteristic curve of $S$ the $1+(n+1)(n+2)/2$ quantities $y, z, p_{rs}$ considered as functions of $x$ satisfy a system $(\Sigma)$ of $3+n(n+1)/2$ ordinary differential equations [which will not be reproduced here]. Any system of functions $y, z, p_{rs}$ of $x$ satisfying $(\Sigma)$ and (1) is called a "characteristic strip $C$ of order $n$." As $dF=0$ is an integral of $(\Sigma)$, condition (1) only has to be postulated at one point of $C$. The manifold of characteristic strips depends on $n-1$ arbitrary functions of one variable. The phrase "an integral surface $S$ contains a characteristic strip $C$" has an obvious meaning.

The author's main result is that every characteristic strip $C$ is contained in infinitely many integral surfaces. (The quantities defining $C$ are assumed to have second derivatives satisfying a Lipschitz condition). Then for every characteristic curve lying on an integral surface $S$ there are infinitely many other integral surfaces having contact of order $n$ with $S$ along that curve. The existence of these surfaces is proved by reduction to a canonical system of partial differential equations of a type previously solved by the author [see the preceding review].

More precisely it turns out that the manifold of integral surfaces containing a given characteristic strip $C$ belonging to a root $\rho$ of (2) depends on an arbitrary function of one variable. For this purpose the author introduces the expression

$$(3)\qquad T=P_{n,0}p_{n-1,1}+[-P_{n,0}\rho'+P_{n-1,1}]p_{n-2,2}+\cdots +[(-1)^{n-1}P_{n,0}\rho'^{n-1}+(-1)^{n-2}P_{n-1,1}\rho'^{n-2}+\cdots +P_{1,n-1}]p_{0,n}.$$

Let $x_0$ be a given point of $C$. Then for an arbitrary function $T=T(y)$ (which reduces at $x_0$ to the value of $T$ for $C$ at $x_0$) there is exactly one integral surface $S$ containing $C$ and a characteristic strip $C'\neq C$ through $x_0$, along which the expression (3) reduces to the given $T(y)$.

For $n=2$ the strip $C'$ can be found directly from the knowledge of $T(y)$ by solving the ordinary differential equations $(\Sigma)$. In this case one can also determine $S$ for prescribed characteristic strips $C$ and $C'$, which have second order contact at one point $x_0$. For $n>2$ there does not exist in general an $S$ through given $C$ and $C'$, and $C'$ can only be found by first determining $S$ from the canonical system of partial differential equations.

In the case of quasi-linear equations, where $F$ is linear in the derivatives of highest order, analogous theorems hold for the characteristic strips of order $n-1$. *F. John.*

**Cinquini-Cibrario, Maria. Sopra la teoria delle caratteristiche per le equazioni di ordine $n$ di tipo iperbolico non lineari.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 10(79), 147–154 (1946).

This note contains a summary of the paper reviewed above. *F. John* (New York, N. Y.).

**Salehov, G. S. A generalization of the formulas of d'Alembert and Poisson.** Uspehi Matem. Nauk (N.S.) 2, no. 4(20), 175–182 (1947). (Russian)

Consider the initial value problem $\partial^n u/\partial t^n=\partial^n u/\partial x^n$, $\partial^k u/\partial t^k|_{t=0}=\varphi_k(x)$ $(k=0, 1, \cdots, n-1)$. The formula

$$(*)\quad u(t, x)=n^{-1}\sum_{l=0}^{n-1}\left\{\varphi_0(\epsilon^l t+x)+\sum_{k=1}^{n-1}\int_0^t\frac{(t-\xi)^{k-1}}{(k-1)!}\varphi_k(\epsilon^l\xi+x)d\xi\right\},$$

where $\epsilon$ is a primitive $n$th root of unity, generalizes d'Alembert's formula

$$u(t, x)=\tfrac12\{\varphi_0(x+t)+\varphi_0(x-t)\}+\tfrac12\int_{x-t}^{x+t}\varphi_1(\xi)d\xi,$$

for the case $n=2$. Formula (*) is a special case of a result given in the first section of the paper. The last section is concerned with integral formulas for solutions of $\partial^n v/\partial t^n=\partial^{2n}v/\partial t^{2n}$ which possess properties analogous to Poisson's integral formula

$$v(t, x)=\int_0^b t^{-\frac{1}{2}}\varphi(\xi)e^{-\frac{1}{4}(x-\xi)^2/t}d\xi,$$

for the heat equation ($n=1$). These results had been announced previously [Uspehi Matem. Nauk (N.S.) **2**, no. 2(18), 226–228 (1947); these Rev. **10**, 44]. *J. B. Dias.*

## *Functional Analysis*

**Kantorovič, L. V. Functional analysis and applied mathematics.** Uspehi Matem. Nauk (N.S.) **3**, no. 6(28), 89–185 (1948). (Russian)

The object of the present work, according to the author, is to show that the ideas and methods of functional analysis can be used for the development of effective, practical algorithms for the explicit solutions of practical problems with just as much success as that with which they have been used for the theoretical study of these problems. He further states that in some cases the explicit results obtained from the more general point of view prove to be more complete and precise than those obtained for special cases of the problem. The complete work is to consist of six chapters. The present paper contains only the first four chapters. The first chapter is of an introductory nature. In it are given the fundamental concepts and operations in Banach spaces. Here are also given general theorems of functional analysis which are required for the proofs of the convergence of the processes of successive approximations and iterations. The second chapter contains the general theory of the methods of approximate solutions for linear problems. It is based on approximate equations in Banach spaces. The third chapter is devoted to the method of steepest descent as it is applicable in Hilbert space. Chapter four contains a study of Newton's algorithm for the solution of nonlinear equations in functional analysis. *H. P. Thielman.*

**Ganapathy Iyer, V. On the space of integral functions. I.** J. Indian Math. Soc. (N.S.) **12**, 13–30 (1948).

Let $S_1$ denote the set of all complex sequences $\alpha=(\alpha(0), \alpha(1), \cdots)$ such that $\limsup|\alpha(n)|^{1/n}<\infty$, and set $\|\alpha\|=\sup_n\{|\alpha(0)|, |\alpha(n)|^{1/n}\}$. Then $S_1$ is a linear space, and is a complete metric space under $\|\ \|$; it is not a topological linear space, for the operation of multiplication by a complex scalar is not continuous in both factors. Let $S_0$ be the largest subspace of $S_1$ which is a topological linear space. Then, $\alpha\epsilon S_0$ if and only if $\lim|\alpha(n)|^{1/n}=0$; $S_0$ is separable and, being a closed subspace of $S_1$, is complete. It is isomorphic to the linear space $\Gamma$ of all entire functions, topologised by uniform convergence on every compact set. This topology is not normable. There is a natural algebraic isomorphism of $S_1$ and $\bar{\Gamma}$, the linear space of all continuous functionals on $\Gamma$, where $f\epsilon\bar{\Gamma}$ corresponds to $\gamma\epsilon S_1$ by means of $f(\alpha)=\sum\alpha(n)\gamma(n)$. The weak topology induced on $\Gamma$ by $\bar{\Gamma}$ is the same as the original one. In addition $\bar{\Gamma}$ inherits a topology from $S_1$ which is not the same as the weak topology induced on $\bar{\Gamma}$ by $\Gamma$. The author concludes with a discussion of certain familiar classes of entire functions. The sets of functions of finite order, of functions having a set of preassigned zeros, of functions which omit a selected value, and of all periodic functions are each sets of first category in $\Gamma$. *R. C. Buck* (Providence, R. I.).

**Berri, R. An investigation of the cone of positive elements in a partially ordered space.** Mat. Sbornik N.S. **23(65)**, 419–440 (1948). (Russian)

Let $E$ be an arbitrary partially ordered linear space which is a lattice under its partial ordering, and let $T$ be a convex subset of $E$. If $x_0$ and $y$ are arbitrary elements of $E$, the set of all elements $x_0+ty$ $(-\infty<t<+\infty)$ is a straight line of $E$. Let $\{t\}$ represent the set of all numbers $t$ such that $x_0+ty\epsilon T$. If 0 is an interior point of $\{t\}$ for all $y\epsilon E$, $x_0$ is said to be an interior point of $T$; if 0 is a boundary point of $\{t\}$ for some $y\epsilon E$, then $x_0$ is said to be a boundary point of $T$; if $0\bar{\epsilon}\{t\}^-$ for all $y\epsilon E$, then $x_0$ is said to be an exterior point of $T$. A convex set $T$ such that every line through every point of $T$ contains points exterior to $T$ is said to be bounded; a convex set containing interior points is said to be dimensionally complete; a convex set containing all of its boundary points is said to be closed. A functional on $E$ is said to be linear if it is additive and distributive.

The author considers in $E$ the cone $K$ consisting of strictly positive elements, making the following assumptions: (1) $K$, together with the point $0\epsilon E$, is a closed set containing interior points; (2) there is a set $F$ of linear functionals on $E$ such that, for every boundary point $x$ of $K$, there is a functional $f\epsilon F$ such that $f(x)=0$ and $f(y)\geqq 0$ for all $y\epsilon K$ (i.e., the hyperplane $f=0$ is to be a hyperplane of support for $K$); (3) there exists a functional of $F$ which is strictly positive throughout $K$. No mention is made of the conditions under which such a family $F$ exists.

The cone $K$ is said to satisfy the condition of regularity with respect to the family $F$ of linear functionals if: (1) there exists a boundary element $g$ of $K$ for which a unique hyperplane of support of $K$ exists containing $g$ and defined by a functional $f_g$ in $F$; (2) there exists a boundary element $e$ of $K$ distinct from $g$ such that, for some hyperplane of support of $K$, going through $e$ and defined by a functional $f_e\epsilon F$, the intersection of the set of $x\epsilon E$ for which $f_g(x)=0$, the set of $x\epsilon E$ for which $f_e(x)=0$ and $K$ is void.

For any element $x$ of $E$, the set of elements $tx$ $(0<t<+\infty)$ is said to be a ray, and is denoted by $0x$. A ray of the cone $K$ defined by a boundary element $e$ is said to be a rib of $K$ if there exists a hyperplane of support of $K$ defined by a functional $f\epsilon F$ such that $K$ is the convex hull of the ray $0e$ and the intersection of the set of $x\epsilon E$ for which $f(x)=0$ and $K$.

The paper is principally devoted to showing that, under the hypotheses made on the cone $K$, the cone $K$ has a rib if and only if it satisfies the condition of regularity. The proofs are of an elementary geometric character. It is to be noted that no topology is assumed for the space $E$. In case $E$ is a separable Banach space, and $F$ is taken to be the family of all continuous linear functionals on $E$, a theorem of Mazur [Studia Math. **4**, 70–84 (1933)] shows that the first condition of regularity is met; and if $E$ is regular, the author states that the second condition is satisfied for any bounded convex set. A final section of the paper deals with the structure of $K$ in case $E$ is a separable but not necessarily regular Banach space. Here it is shown that $K$ has a simplicial structure which is the direct analogue of the structure of $K$ in case $E$ is $n$-dimensional. (Here $K$ is the convex hull of $n$ linearly independent rays.)

*E. Hewitt* (Seattle, Wash.).

**Kawada, Yukiyosi. On a generalization of the theory of ratio of Euclid.** Jap. J. Math. 19, 375–384 (1947).

The author's object is to extend Euclid's theory of ratio of elements to the case where the elements are partially ordered. For this purpose the elements are so restricted that they may be considered as the positive elements in a partially ordered real linear vector space. The restrictions also imply that this vector space is conditionally complete and contains a dominating positive element $e$ (this means that for every $x$ there is some positive integer $n$ such that $x < ne$). A calculus of ratios $u:v$ is then developed for positive elements $u, v, \cdots$ which dominate $e$, that is, $e < mu$ for some integer $m$, etc. The results are used to construct a product $xy$ for arbitrary positive elements.

The conditions assumed by the author do not seem to be entirely adequate, particularly in connection with his theorems 2, 4 and the (unstated) law of cancellation that from $x+y=x+z$ it follows that $y=z$. The results of the present paper are also consequences of the more general theorems of Nakano [Proc. Phys.-Math. Soc. Japan (3) 23, 485–511 (1941); these Rev. 3, 210]. *I. Halperin.*

**Dixmier, Jacques. Mesure de Haar et trace d'un opérateur.** C. R. Acad. Sci. Paris 228, 152–154 (1949).

Let $E$ be a real partially ordered Banach space with a "unit" element 1. Let $G$ be a group of linear order preserving isometries of $E$. The author states a theorem asserting that two additional hypotheses on the system $E$, $G$ imply the existence of a unique order-preserving linear transformation of $E$ into $E$ such that $T(1)=1$ and such that for all $x$ in $E$ and all $s$ in $G$ it is true that $T(sx)=sT(x)=T(x)$. Let $E$ be the space of all real-valued continuous functions on a compact group $K$ and let $G$ be the group of all transformations of the form $x(t)\rightarrow x(st)$, where $s\epsilon K$. Then the two hypotheses are satisfied and $T(x)$ is the constant function identically equal to the Haar integral of $x$. Let $E$ be the space of all self-adjoint linear operators in a weakly closed ring with a unit $M$. Let $G$ be the group of all transformations of the form $X\rightarrow UXU^{-1}$, where $U$ is unitary and in $M$. The author asserts that hypothesis one is always satisfied and that hypothesis two is satisfied if and only if whenever $U\epsilon M$ and $UU^*=1$ then also $U^*U=1$. In particular, hypothesis two is satisfied whenever $M$ is a factor in a finite case. Then $T(X)$ is a constant multiple of the identity and the constant is the relative trace (in the sense of Murray and von Neumann) of $X$. No proofs are given.

*G. W. Mackey* (Cambridge, Mass.).

**Fréchet, Maurice. La notion de différentielle sur un groupe Abélien.** Portugaliae Math. 7, 59–72 (1948).

Let $t$ and $T$ be topological Abelian groups, written additively. Let $F$ be a function with values in $T$, defined throughout a neighborhood of a point $x_0\epsilon t$. By a linear function $L$ on $t$ to $T$ is meant a function which is defined and continuous throughout $t$, with values in $T$, and additive: $L(x+y)=L(x)+L(y)$. The guiding principle of Fréchet's definition of a differential is that the differential of $F$ at $x=x_0$, with increment $y$, is to be a linear function $L(y)$ which is in a certain sense a first order approximation to $F(x_0+y)-F(x_0)$. He first reformulates his definition for the case in which $t$ and $T$ are normed linear spaces in such a way that it continues to have meaning in topological Abelian groups. The definition then becomes: $L$ is the differential of $F$ at $x=x_0$ if there exists a neighborhood $W$ of $0\epsilon t$ such that to each neighborhood $V$ of $0\epsilon T$ corresponds a neighborhood $U$ of $0\epsilon t$ with the property that, for each positive integer $n$, if $y\epsilon U$ and $ny\epsilon W$ then $n[F(x_0+y)-F(x_0)-L(y)]\epsilon V$. This definition has the consequences: (1) $F$ is continuous at $x=x_0$ if it is differentiable there; (2) the composite function rule for differentials is valid. To prove that a differentiable function has a unique differential, Fréchet follows A. D. Michal in assuming that $t$ has the "Archimedean" property: given $y\epsilon t$ and a neighborhood $S$ of $0\epsilon t$, there exists $x\epsilon S$ and a positive integer $n$ such that $nx=y$. Functions of several variables are also considered. Reference is made to work of A. D. Michal [Revista Ci., Lima 47, 389–422 (1945); Math. Mag. 21, 80–90 (1947); these Rev. 7, 308; 9, 358]. Fréchet's present definition is given independently by Michal in the second paper cited [the text of an address delivered in 1941].

*A. E. Taylor* (Los Angeles, Calif.).

**Šifrin, I. A. On the extrema of compound functionals.** Učenye Zapiski Kazan. Univ. 101, kn. 3, 19–21 (1941). (Russian)

Let $J_0, \cdots, J_n$ be functionals on the Banach space $C^2$, and let $\varphi_i$ be real-valued functions on $(n+1)$-space $(i=0, \cdots, k<n)$. A minimum for $\varphi_0(J_0, \cdots, J_n)$ is sought subject to side-conditions $\varphi_i(J_0, \cdots, J_n)=0$ $(i=1, \cdots, k)$. It is indicated that the problem is related to an isoperimetric problem. *E. J. McShane* (Charlottesville, Va.).

**Beurling, Arne. On two problems concerning linear transformations in Hilbert space.** Acta Math. 81, 17 pp. (1948).

Let $T$ be a bounded linear transformation in Hilbert space $H$ and $T^*$ its adjoint. Denote by $C_f$ and $C_g^*$ the subspaces spanned by $\{T^nf\}_0^\infty$ and $\{T^{*n}g\}_0^\infty$, respectively. The "extinction problem" is to characterize those transformations which have an "extinction theorem," i.e., for which each $C_f$ $(f\neq 0)$ contains at least one eigenelement $\varphi\neq 0$. The "closure problem" is to characterize the elements $g$ for which $C_g^*=H$, by the behaviour of $(\varphi, g)$ when $\varphi$ runs through the set $\Phi$ of all eigenelements of $T$. The necessary condition $(\varphi, g)\neq 0$ is obvious. If this condition is also sufficient, then $T$ is said to possess a "Wiener closure theorem." It is obvious that, under the condition: (A) the set $\Phi$ is fundamental in $H$, the extinction theorem implies the Wiener closure theorem. However, it is difficult, in general nontrivial cases, to decide, whether these two theorems are true and if they are equivalent.

If $T$ is isometric and (A) holds, then the extinction theorem is a simple consequence of von Neumann's ergodic theorem. Making use of the generalization of this theorem to uniformly convex Banach spaces, due to G. Birkhoff and F. Riesz, the following result may be obtained. If $T$ is a linear isometric transformation of such a space and if (A′) $\varphi\perp f$ for all $\varphi\epsilon\Phi$ implies $f=0$, then $T$ has an extinction theorem. The notation $\varphi\perp f$ means that $\|\varphi+cf\|\geqq\|\varphi\|$ for every complex number $c$.

The major part of the paper deals with transformations $T$ of $H$ for which $\|Tf\|\leqq f$, $\|T^*f\|=\|f\|$ and $\lim_{n\to\infty}\|T^nf\|=0$ for all $f\epsilon H$, and which satisfy, besides (A), also the condition that at least one eigenvalue is simple. It is shown that such a transformation is always isomorphic with the transformation $Tf(z)=[f(z)-f(0)]/z$, defined in the Hilbert space $H$ of the Taylor series $f(z)=\sum_0^\infty a_nz^n$ with $\sum|a_n|^2<\infty$, provided with the inner product

$$(f_1, f_2)=(2\pi)^{-1}\int_0^{2\pi} f_1(e^{it})\bar{f}_2(e^{it})dt=\sum a_{1n}\bar{a}_{2n},$$

where $f(e^{it})$ denotes the (almost everywhere existing) limit $\lim_{r\to 1} f(re^{it})$. The eigenvalues of this transformation are all simple and fill the interior of the unit circle; the eigenelement corresponding to $\lambda$ is $\varphi_\lambda=(1-\lambda z)^{-1}$. The adjoint of $T$ is $T^*f(z)=zf(z)$.

For this transformation, the closure problem reads as follows. For which $f(z)$ is the set $\{z^n f(z)\}_0^\infty$ fundamental in $H$? The "Wiener criterion" $(\varphi_\lambda, f)\neq 0$ yields $f(z)\neq 0$ for $|z|<1$, but it is found that this condition is not sufficient. The adequate condition turns out to be $\delta(f)=0$, where $\delta(f)=(2\pi)^{-1}\int_0^{2\pi}\log|f(e^{it})/f(0)|dt$ if $f(0)\neq 0$ and $\delta(f)=+\infty$ if $f(0)=0$. The proof is based on the decomposition lemma of F. Riesz and R. Nevanlinna [cf. Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, chap. VII], which states that a function $f(z)$, holomorphic in the unit circle and belonging to a Hardy class $H_p$ $(p>0)$, may be decomposed in the form $f_0(z)f_1(z)$, where

$$f_1(z)=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{it})|\frac{e^{it}+z}{e^{it}-z}dt+ia\right\},$$

$a$ being the argument of the first nonvanishing Taylor coefficient of $f(z)$, and

$$f_0(z)=\prod\frac{a_\nu-z}{1-z\bar{a}_\nu}\frac{\bar{a}_\nu}{|a_\nu|}\exp\left\{-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}d\alpha(t)\right\};$$

here $\alpha(t)$ is a real nondecreasing bounded function whose points of increase form a set of at most measure 0 and one has $|f_0(z)|\leqq 1$ for $|z|<1$ and $|f_0(e^{it})|=1$ almost everywhere.

The author proves the fact that $f_0(z)$ (the "inner factor" of $f(z)$), is precisely the orthogonal projection of the element $e(z)\equiv 1$ on the subspace $C_f^*$, multiplied by $(1-d^2)^{\frac{1}{2}}$, where $d=d(f)$ denotes the distance of $e(z)$ from $C_f^*$, thus characterizing the inner factor by an extremal property. Moreover, he proves that $d^2=1-e^{-2\delta(f)}$; as $d=0$ is both necessary and sufficient in order that $C_f^*=H$, this yields that the condition $\delta(f)=0$ is adequate. In any case, the subspace $C_f^*$ consists precisely of the functions for which $g_0/f_0$ is bounded in the unit circle ($f_0$ is a "divisor" of $g_0$).

The closure problem may be generalized by considering an arbitrary subspace $C^*$ such that $T^*C^*\subseteq C^*$. It is shown that such a $C^*$ can always be generated by an inner function in the sense $C^*=C_{f_0}^*$.

For the extinction problem, the following result is obtained. Any invariant subspace $C\neq 0$ of $H$, i.e., one for which $TC\subseteq C$, contains either at least one eigenelement $\varphi_\lambda(z)$ or, otherwise, a function of the form

$$1-\exp\left\{-\int_0^{2\pi}(1-ze^{it})^{-1}d\mu(t)\right\}\neq 0,$$

where $\mu(t)$ is a nondecreasing bounded function whose points of increase form a set of at most zero measure.

*B. de Sz. Nagy* (Szeged).

**Aronszajn, N. Rayleigh-Ritz and A. Weinstein methods for approximation of eigenvalues. I. Operators in a Hilbert space.** Proc. Nat. Acad. Sci. U. S. A. **34**, 474–480 (1948).

The author considers in a Hilbert space $\mathfrak{H}$, with the scalar product $(x, y)$, a symmetric and completely continuous operator $H$. For any closed subspace $\mathfrak{L}\subset\mathfrak{H}$, he denotes the corresponding projection by $P$. The operator $L=PH$, considered in the subspace $\mathfrak{L}$, is called "the part of $H$ in $\mathfrak{L}$." The eigenvalues, eigenvectors, resolvent operator, etc., of $L$ (in $L$) are called the eigenvalues, etc., of $H$ in $\mathfrak{L}$. An eigenvector $u$ and the corresponding eigenvalue $\lambda$ of $H$ in $\mathfrak{L}$ satisfy the equation $Hu-\lambda u=p$ with $p\perp\mathfrak{L}$. The positive eigenvalues $\lambda_0\geqq\lambda_1\geqq\cdots\to 0$ are defined by maximum-minimum problems [Courant], and in an analogous way the negative eigenvalues $\mu_0\leqq\mu_1\leqq\cdots\to 0$. The author proves the following theorems: (I) If $\mathfrak{L}'\subset\mathfrak{L}$, $\mathfrak{L}''=\mathfrak{L}\ominus\mathfrak{L}'$, we have the inequalities $\lambda_{i+j}+\mu_0\leqq\lambda_i'+\lambda_j''$, $\mu_{i+j}+\lambda_0\geqq\mu_i'+\mu_j''$ for $i, j=0, 1, 2, \cdots$. (II) If a sequence of subspaces $\mathfrak{L}^{(n)}$ converges to a subspace $\mathfrak{L}$: $\mathfrak{L}^{(n)}\to\mathfrak{L}$, then $\lambda_k^{(n)}\to\lambda_k$, $\mu_k^{(n)}\to\mu_k$.

Furthermore, the author considers two subspaces, $\mathfrak{L}'\subset\mathfrak{L}$, with $\mathfrak{L}\ominus\mathfrak{L}'$ $n$-dimensional, a system of $n$ vectors $p_1, \cdots, p_n$ generating the subspace $\mathfrak{L}\ominus\mathfrak{L}'$, the resolvent operator $R_\lambda$ of $H$ in $\mathfrak{L}$ and $u_m(\lambda)=R_\lambda p_m$, $m=1, \cdots, n$ ($\lambda$ any complex number). He denotes by $W(\xi)$ (Weinstein's determinant) the $n$th order determinant $W(\xi)=\det\{(u_m(\xi), p_k)\}$, by $\Gamma=\Gamma\{p_k\}$ the Gram determinant $\Gamma\{p_k\}=\det\{(p_m, p_k)\}$ and by $D(\xi)$ the $n$th order determinant

$$D(\xi)=\det\{(Hw_k+Hp_k-\xi p_k, p_i)\},$$

and proves two theorems about the analytic character of $\phi(\xi)=W(\xi)/\Gamma\{p_k\}$ and $\psi(\xi)=D(\xi)/\Gamma\{p_k\}$. On the basis of these theorems, he develops a generalization of Ritz's and Weinstein's methods [Weinstein, Etude des spectres des équations aux dérivées partielles, Mémor. Sci. Math., no. 88, Gauthier-Villars, Paris, 1937] for the computation of the eigenvalues of the operator $H$ in a given subspace $L$.

*P. Funk* (Vienna).

**Aronszajn, N. The Rayleigh-Ritz and the Weinstein methods for approximation of eigenvalues. II. Differential operators.** Proc. Nat. Acad. Sci. U. S. A. **34**, 594–601 (1948).

Application of the considerations of the paper reviewed above to differential operators, particularly of the type considered in Weinstein's memoir. *P. Funk* (Vienna).

**Van Hove, L. Un prolongement de l'espace fonctionnel de Hilbert.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 604–616 (1948).

Un sous-espace $L$, dense dans un espace de Hilbert $H$, permet de définir un prolongement $L^0$ de $H$: $X\epsilon L^0$ est une forme antilinéaire $X(x)$ définie sur $L$, c'est-à-dire $X(a_1x_1+a_2x_2)=\bar{a}_1X(x_1)+\bar{a}_2X(x_2)$ si $a_1$ et $a_2$ sont des scalaires complexes, $x_1$ et $x_2\epsilon L$. L'espace $H$ est bien immergé dans $L^0$, car à $y\epsilon H$ correspond $Y\epsilon L^0$ définie par $Y(x)=\langle y, x\rangle$ pour $x\epsilon L$. Si $A$ est un opérateur linéaire défini dans une partie de $H$, d'adjoint $A^*$, tel que $(A^*)^{-1}(L)$ soit dense dans $H$, on peut prolonger l'opérateur $A$ à $L^0$ par la formule $AX(x)=X(A^*x)$; $x\epsilon(A^*)^{-1}(L)$, $A^*(x)\epsilon L$, $X\epsilon L^0$, $AX\epsilon((A^*)^{-1}(L))^0$. Si $H$ est l'espace des fonctions de carré sommable sur la droite, $L$ un espace de fonctions indéfiniment dérivables convenable, on obtient pour $X$ des "distributions" [L. Schwartz, Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 21 (1945), 57–74 (1946); ces Rev. **8**, 264]. On peut alors, suivant les cas, prendre pour opérateur $A$ la dérivation, la transformation de Fourier, le prolongement analytique. *L. Schwartz* (Nancy).

**Mikusiński, Jan G. Sur la méthode de généralisation de M. Laurent Schwartz et sur la convergence faible.** Fund. Math. **35**, 235–239 (1948).

L'espace vectoriel des distributions [Schwartz, Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 21 (1945), 57–74 (1946); ces Rev. **8**, 264] peut être défini comme le complété de l'espace des fonctions pour une topologie convenable.

L'auteur étudie ce procédé général de formation d'êtres mathématiques nouveaux par la notion de convergence faible et de complétion. Il semble m'attribuer la paternité de cette méthode de complétion, qui remonte à Cantor (définition des nombres réels). *L. Schwartz* (Nancy).

**Ryll-Nardzewski, Czesław. Une remarque sur la convergence faible.** Fund. Math. **35**, 240–241 (1948).

Remarque sur l'article de Mikusiński [analysé ci-dessus]. *L. Schwartz* (Nancy).

**Kac, M. On distributions of certain Wiener functionals.** Trans. Amer. Math. Soc. **65**, 1–13 (1949).

The author develops a method for calculating the distribution function $\sigma(\alpha;t)$ for the functional $\int_0^t V(x(\tau))d\tau$, where $x(t)$ varies over the space $C$ of continuous functions on $0\leqq t<\infty$ which vanish at the origin. In terms of the given function $V(\lambda)$ (which must satisfy certain mild restrictions) the Green's function $\psi(\lambda)=\psi(\lambda, s, u)$ is defined by the differential equation $\frac{1}{2}\psi''-(s+uV(\lambda))\psi=0$ subject to the conditions $\psi(\lambda)\rightarrow 0$ as $\lambda\rightarrow\pm\infty$, $\psi'(\lambda)$ is bounded, and $\psi'(0+)-\psi'(0-)=-2$. Then the double Laplace transform of the distribution function $\sigma$ (expressed here both in Kac's notation and in the notation of Wiener integrals) is given by the equation

$$\int_0^\infty\int_0^\infty \exp(-u\alpha-st)d_\alpha\sigma(\alpha;t)dt$$

$$=\int_0^\infty\int_C^W \exp\left\{-st-u\int_0^t V(x(\tau))d\tau\right\}d_Wxdt$$

$$=\int_{-\infty}^\infty \psi(\lambda, s, u)d\lambda.$$

*R. H. Cameron* (Minneapolis, Minn.).

## *Calculus of Variations*

**de Franchis, Franco. Sur les trajectoires des problèmes variationnels.** C. R. Acad. Sci. Paris **227**, 1013–1015 (1948).

This short note contains a remark on the relation of differential equations to variational problems. *L. M. Graves* (Chicago, Ill.).

**Šifrin, I. A. On a variational problem.** Učenye Zapiski Kazan. Univ. **101**, kn. 3, 13–18 (1941). (Russian)

The problem is to minimize

$$I(z)=\int_a^b F(x, z_1, \cdots, z_m, z_1', \cdots, z_m')dx$$

in a class of functions $z$ which satisfy certain end conditions and also give assigned value at $b$ to the solution $y$ of the equation (*) $d^ny/dx^n=G(x, z_1, \cdots, z_m, z_1', \cdots, z_m')$, the values of $y, \cdots, y^{(n-1)}$ at $a$ being also assigned. The Lagrange multiplier rule is obtained. The author overlooks the fact that the equation (*) and the end-conditions on $y, \cdots, y^{(n-1)}$ are equivalent to isoperimetric conditions, with integrands of the form $(b-x)^iG(x, z_1, \cdots, z_m')$. *E. J. McShane* (Charlottesville, Va.).

**Cinquini, Silvio. Sopra le condizioni necessarie per la semicontinuità degli integrali dei problemi variazionali in forma parametrica di ordine superiore.** Ann. Scuola Norm. Super. Pisa (2) **14** (1945), 1–19 (1948).

For an integral $J(C)=\int F(x, y, x', y', \theta')dt$ depending on the curvature $\theta'$ of the plane curve $C$, the author proves the expected necessary condition for lower semicontinuity of $J(C)$ at a curve $C_0$, namely, that

$$\mathcal{E}(x_0(s), y_0(s), x_0'(s), y_0'(s), \theta_0'(s), \theta')\geqq 0$$

for all $\theta'$ and for almost all values of $s$ for which $(x_0(s), y_0(s))$ is interior to the domain $A$ where admissible curves must lie and for which the curvature $\theta_0'(s)$ is finite. As one might anticipate, the proof is rather long and complicated. The paper concludes with an example of a minimizing curve for a certain integral, having a point at which the curvature exists and is finite, but the Weierstrass condition fails. Obviously the curvature is discontinuous at this point. *L. M. Graves* (Chicago, Ill.).

**Damköhler, Wilhelm. Über die Äquivalenz indefiniter mit definiten isoperimetrischen Variationsproblemen.** Math. Ann. **120**, 297–306 (1948).

This is a continuation of a paper by Damköhler and Hopf [same vol., 12–20 (1947); these Rev. **9**, 242], and takes up the case when there are isoperimetric side conditions of the form $\int_\gamma G_\alpha(x, \dot{x})dt=L_\alpha$ $(\alpha=1, \cdots, m)$. The "definiteness character" of an integral $\int F(x, \dot{x})dt$ relative to these side conditions is defined by the formula $\chi=\inf\{\int_\gamma Fdt/\Gamma(\int_\gamma Gdt)\}$ taken over the class of all closed rectifiable curves in the region, where $\Gamma(y_1, \cdots, y_m)$ is a fixed convex function of $m$ variables, positive except for $y_1=\cdots=y_m=0$, and satisfying the homogeneity condition $\Gamma(hy)=|h|\Gamma(y)$. The principal theorem states that the finiteness of $\chi$ is necessary and sufficient for the existence of a function $f(x, y)$ satisfying a Lipschitz condition in the small, such that every curve $\gamma$ joining $x_1$ and $x_2$ and making $\int_\gamma G_\alpha(x, \dot{x})dt=L_\alpha$ also satisfies $\int_\gamma F(x, \dot{x})dt+f(x_2, L)-f(x_1, 0)\geqq 0$. The proof uses the familiar process of enlarging the space $x$ by the adjunction of the additional variables $y_1, \cdots, y_m$. The paper also includes some theorems giving sufficient conditions that $\chi$ shall be finite. *L. M. Graves* (Chicago, Ill.).

**Sigalov, A. G. On quasiregular double integrals of the calculus of variations.** Mat. Sbornik N.S. **23**(65), 127–158 (1948). (Russian)

If $F(x, y, z, p, q)$ is continuous in all variables, nonnegative and convex in $(p, q)$ for each $(x, y, z)$, the integral $\iint_D F(x, y, f(x, y), f_x(x, y), f_y(x, y))dxdy$ (where $D$ is the square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$) defines a functional on the class of surfaces absolutely continuous in the sense of Torelli. This functional is lower semi-continuous on the class of surfaces mentioned. On each surface of this class the value of the integral is the limit of its values on a sequence of polyhedra converging to the surface. It is not assumed that $F$ has continuous derivatives; however, the semicontinuity theorem with $F$ assumed convex is known to follow without great difficulty from the case in which $F$ has continuous derivatives and is positive quasi-regular. The proofs do not require a knowledge of the theory of Lebesgue area. The results can also be obtained by specializing to nonparametric form certain known theorems on integrals in parametric form [T. Radó, Trans. Amer. Math. Soc. **51**, 336–361 (1942); also W. Scott, Bull. Amer. Math. Soc. **48**, 763–767; these Rev. **3**, 229; **4**, 155]. *E. J. McShane.*

## *Theory of Probability*

✱**Copeland, Arthur H. A postulational characterization of statistics.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 51–61. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

The author presents a long list of postulates for a set $R$ of elements called "variates." His postulates imply that $R$ is a commutative algebra, with unit, over the field of complex numbers. He also postulates the existence of an operation $x \subset y$, defined for every $x$ in $R$ and for every nonzero idempotent $y$, which he asserts is to be interpreted as von Mises' operation of "Auswahl." Since the paper does not describe any model satisfying the postulates, it is not clear to the reviewer that, as the author asserts, "the system $R$ constitutes an adequate basis for statistics," or what advantages this system has over Kolmogoroff's definition of a random variable as a measurable function on a measure space. *P. R. Halmos* (Chicago, Ill.).

**Ţiţeica, Şerban. Sur un problème de probabilités.** Bull. Math. Phys. Éc. Polytech. Bucarest **10** (1938–39), 57–64 (1940).

The following occupancy problem, said to be needed in a study of bacteriophages, is treated. Let $N$ objects be distributed into $P$ boxes with probability $P^{-1}$ that any object falls in any box; what is the probability $q$ that no box contains more than $n-1$ objects? From the exponential generating function $\sum P^N q_N z^N/N! = [f(z)]^P$ with $f(z) = 1+z+z^2/2!+\cdots+z^{n-1}/(n-1)!$, the author finds, by using the method of steepest descent on the Cauchy contour integral for the $N$th derivative of the left hand side, the approximation

$$q = \{F_1(x)/F_2(x)\}^{\frac{1}{2}}[f(x) \exp \{F_1(\log (F_1/x)-1)\}]^P$$

with $F_1(x) = xf'(x)/f(x)$, $F_2(x) = xF_1'(x)$, and $x$ determined by $F_1(x) = NP^{-1}$. For $NP^{-1}$ small this becomes $q = [f(NP^{-1}) \exp (-NP^{-1})]^P$, that is, the $P$th power of the Poisson summation with mean $NP^{-1}$. *J. Riordan.*

**Bendersky, L. Sur quelques problèmes concernant les épreuves répétées.** Bull. Sci. Math. (2) **72**, 99–107 (1948).

The author computes the expectation of the number of iterations of length $k$ in $n$ independent trials with 2 possible outcomes and related expectations. *J. L. Doob.*

✱**Pólya, G. Remarks on computing the probability integral in one and two dimensions.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 63–78. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

Let $g(x) = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$, $G(x) = \int_0^x g(t)dt$,

$$L(a, a'; b, b'; r)$$
$$= \int_a^b \int_{a'}^{b'} [2\pi(1-r^2)]^{-\frac{1}{2}} g\{[x^2-2rxx'+x'^2/(1-r^2)]^{\frac{1}{2}}\} dx'dx,$$
$$M(h, k; r) = L(h, k; \infty, \infty, r).$$

The main results are as follows. (1) $2G(x) < (1-e^{-2x^2/\pi})^{\frac{1}{2}}$, $x>0$, also given by J. D. Williams [Ann. Math. Statistics **17**, 363–365 (1946); these Rev. **8**, 56]. (2) Three unusual proofs that $2G(\infty) = 1$, one by the method of residues. (3) For $0<r<1$, $rh-k>0$, $M(h, k; r) < \frac{1}{2}-G(h)$,

$$M(h, k; r) > \tfrac{1}{2}-G(h) - \frac{1-r^2}{rh-k} g(k) \left\{ \tfrac{1}{2} - G\left( \frac{h-rk}{(1-r^2)^{\frac{1}{2}}} \right) \right\}.$$

(4) An approximate formula for $L(a, a'; b, b'; r)$ particularly good for $r$ near one. (5) A lemma on enveloping series and a formula for

$$V(h, k) = \int_0^h \int_0^{kx/h} g(x)g(y)dydx$$

which involves a divergent enveloping series. Formulas (3), (4), (5) have been employed to extend the tables of $M(h, k; r)$ given in Pearson's Tables for Statisticians and Biometricians, part II. *L. A. Aroian* (New York, N. Y.).

**Camp, Burton H. Generalization to $N$ dimensions of inequalities of the Tchebycheff type.** Ann. Math. Statistics **19**, 568–574 (1948).

For the probability density $f(X_1, \cdots, X_n)$ of an $n$-dimensional random variable, and any $\lambda \geqq 0$, let $x_\lambda$ be the measure of the set $Q_\lambda$ of those $(X_1, \cdots, X_n)$ for which $f>\lambda$. Then $\lambda = \lambda(x)$ can be uniquely defined and is nonincreasing. The $r$th contour-moment is defined by $\mu_r^* = \int_0^\infty x^r \lambda(x)dx$ [not the author's notation]; let $\sigma^* = \sqrt{\mu_2^*}$, $\alpha_r^* = \mu_{2r}^*/(\sigma^*)^{2r}$. The following theorem is proved: if $x_\lambda = \delta\sigma^*$ and

$$P_\delta = \int_{Q_\lambda} f(X_1, \cdots, X_n)dX_1 \cdots dX_n,$$

then $1-P_\delta \leqq \alpha_{2r}^*/(\frac{1}{2}\delta(2r+1)/r)^{2r}$ for $r \geqq 1$. A number of corollaries, examples and numerical illustrations are given. *Z. W. Birnbaum* (Seattle, Wash.).

**Rényi, Alfred. Simple proof of a theorem of Borel and of the law of the iterated logarithm.** Mat. Tidsskr. B. **1948**, 41–48 (1948).

Let $S_n(x)$ denote the excess of ones over zeros in the first $n$ digits of the dyadic expansion of $x$, $0 \leqq x \leqq 1$. The author proves in a simple manner that, for almost all $x$, $S_n(x) = o(n)$ and $\limsup (2n \log \log n)^{-\frac{1}{2}} S_n(x) = 1$. *J. Wolfowitz* (New York, N. Y.).

**Chung, Kai Lai. Asymptotic distribution of the maximum cumulative sum of independent random variables.** Bull. Amer. Math. Soc. **54**, 1162–1170 (1948).

Let $X_1, X_2, \cdots$ be mutually independent random variables with equal variances and expectations $\mu_1, \mu_2, \cdots$. The asymptotic distribution of $\max_{k \leqq n} \sum_1^k X_j$ is found $(n \to \infty)$, with an estimate of the error. The method used covers the case $\mu_1 = \mu_2 = \cdots$ [for which the asymptotic distribution was found by Erdös and Kac, same Bull. **52**, 292–302 (1946); these Rev. **7**, 459] and the case $\mu_k = cn^{-\frac{1}{2}}$, $k \leqq n$, $c$ constant [for which the asymptotic distribution was found by Wald, same Bull. **53**, 142–153 (1947); these Rev. **8**, 471]. *J. L. Doob* (Urbana, Ill.).

**Sapogov, N. A. On the law of the iterated logarithm for dependent variables.** Doklady Akad. Nauk SSSR (N.S.) **63**, 487–490 (1948). (Russian)

The author shows that under Bernstein's conditions for the validity of the central limit theorem for dependent random variables [Math. Ann. **97**, 1–59 (1926), § 10] the law of the iterated logarithm also holds. He uses a form of the central limit theorem with error estimate found by him in an earlier paper [same Doklady (N.S.) **63**, 353–356 (1948); these Rev. **10**, 310]. *J. L. Doob.*

**Robbins, Herbert. The asymptotic distribution of the sum of a random number of random variables.** Bull. Amer. Math. Soc. **54**, 1151–1161 (1948).

Let $X_1, X_2, \cdots$ be mutually independent random variables with a common distribution function. Let $N$ be a random variable, independent of the $X_j$'s, taking on only positive integral values. Let $Y=X_1+\cdots+X_N$. It is supposed that the distribution of $N$ depends on a parameter $\lambda$, and the limiting distribution of $Y$ $(\lambda\rightarrow\infty)$ is investigated. For example, it is shown that if $N$, normalized as usual in terms of its first and second moments, is asymptotically normal, then $Y$, normalized in the same way, is also asymptotically normal. *J. L. Doob* (Urbana, Ill.).

**Loève, Michel. Sur l'équivalence asymptotique des lois.** C. R. Acad. Sci. Paris **227**, 1335–1337 (1948).

The author states results on functions of bounded variation and their Fourier-Stieltjes transforms containing, for example, the Lévy continuity theorem in the special case when the functions are distribution functions. He then states conditions in terms of the distribution functions of the summands that the distribution functions of the sums $\sum_{j=1}^{n} X_{nj}$ and $\sum_{j=1}^{n} Y_{nj}$ are asympotically equivalent (in a sense made precise in the paper) when $n\rightarrow\infty$. These are stated in such a way that they become necessary in important applications when the summands are mutually independent. *J. L. Doob* (Urbana, Ill.).

***Brockmeyer, E., Halstrøm, H. L., and Jensen, Arne. The life and works of A. K. Erlang.** Trans. Danish Acad. Tech. Sci. **1948**, no. 2, 277 pp. (1948).

This useful compendium contains Erlang's principal papers, together with (1) a biography by Brockmeyer and Halstrøm; (2) a detailed discussion in accordance with the modern methods of probability theory of his classical application of probability theory to telephone engineering (waiting time and related topics) by Jensen; (3) a survey of his mathematical work by Brockmeyer; (4) a survey of his electrotechnical work by Halstrøm. *J. L. Doob.*

**Kendall, David G. On the role of variable generation time in the development of a stochastic birth process.** Biometrika **35**, 316–330 (1948).

An individual has lifetime $\tau$ and then gives way to two individuals, each of which finally gives way to two and so on. The lifetimes are all mutually independent and distributed like $\chi^2$ with $2k$ degrees of freedom. When $k=1$ the resulting process reduces to the birth process studied by Feller [Acta Bioth. Ser. A. **5**, 11–40 (1939); these Rev. **1**, 22], in which the lifetimes have density of distribution $\lambda e^{-\lambda\tau}$ $(\tau>0)$ so that the author's general assumption is equivalent to assuming that individuals pass through $k$ phases each having a lifetime as in Feller's work before giving way to two individuals. The process is studied in terms of these phases, in terms of which it is a special branching process, as discussed by Kolmogoroff and Dmitriev [C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 5–8 (1947); these Rev. **9**, 46]. When $k\rightarrow\infty$ the lifetimes become fixed numbers. The author suggests that values of $k$ around 20 may be applicable to the growth of populations of elementary organisms. Let $n(t)$ be the number of individuals existing at time $t$. The expectation and variance of $n(t)$ are evaluated, and $n(t)$ is investigated for $t\rightarrow\infty$ and $k\rightarrow\infty$.
*J. L. Doob* (Urbana, Ill.).

**Moran, P. A. P. Some theorems on time series. II. The significance of the serial correlation coefficient.** Biometrika **35**, 255–260 (1948).

[For part I see Biometrika **34**, 281–291 (1947); these Rev. **9**, 361.] Let $x_1, \cdots, x_n$ be mutually independent random variables with a common normal distribution. Dixon [Ann. Math. Statistics **15**, 119–144 (1944); these Rev. **6**, 6] found the moments of the sample serial correlation coefficient (cyclic definition). The author shows how to calculate the lower moments of the sample serial correlation coefficient both in the cyclic and noncyclic forms. In the latter case he evaluates the first three moments centering by sample averages and the first four moments centering by true means; the lag 1 is used in both cases.
*J. L. Doob* (Urbana, Ill.).

***Doob, J. L. Time series and harmonic analysis.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 303–343. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

Expository paper. A note added in proof states that it was written in 1945, and that more recent results might make desirable a complete rearrangement of the material. This, however, would not detract from the value of the careful bibliography, or of the treatment of physical applications. *H. Wold* (Uppsala).

***Feller, W. On the theory of stochastic processes, with particular reference to applications.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 403–432. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

The author presents a broad picture of Markov processes (continuous time parameter) in the purely discontinuous case, representing systems which move by jumps from state to state. The discussion covers the Poisson process and its generalizations leading to Markov chains and finally to the general case, for which the transition probability functions satisfy integro-differential equations of a very general type. Existence and uniqueness theorems for these equations were proved by the author in an earlier paper [Trans. Amer. Math. Soc. **48**, 488–515 (1940); these Rev. **2**, 101]. These equations reduce to systems of differential equations in many applications to growth problems, telephone engineering and so on, for which examples are given and solutions (in some cases not hitherto known) found.
*J. L. Doob* (Urbana, Ill.).

**Bass, Jean, et Le Cam, Lucien. Sur certains classes de fonctions aléatoires.** C. R. Acad. Sci. Paris **227**, 1206–1208 (1948).

Let the random variable $X(t)$ of a stochastic process be defined by $X(t)=\int_0^t K(t,s)d\xi(s)$, where $K(t,s)$ is a given function and $\xi(s)$ is the random variable of a process with orthogonal increments, sometimes more closely specified as required in the discussion. Conditions are stated that the $X(t)$ process have derivatives and that it be a Markov process. Various evaluations are made; for example, if $x=X(t)$, $u=X'(t)$ have a joint probability density $f(x,u,t)$, then $\partial f/\partial t+u\partial f/\partial x$ is evaluated. *J. L. Doob.*

**Blanc-Lapierre, André. Remarques sur certaines fonctions aléatoires.** C. R. Acad. Sci. Paris **227**, 1333–1335 (1948).

Let $X(t)$ be defined by $\int_{-\infty}^{\infty} R(t-s)[dn(s)-ads]$, where $dn(s)$ is the number of events that have occurred in time $ds$

in a Poisson process with rate of occurrence $a$. By means of general theorems on the representation of random variables like $X(t)$ and $\prod_j X(t+h_j)$ as Fourier transforms of other processes, the moments of these random variables and thereby the spectral distribution of the corresponding stochastic processes can be computed. *J. L. Doob.*

**Dedebant, G. Sur le calcul aléatoire.** Anais Fac. Ci. Pôrto 32, 5–48, 65–112, 129–176, 193–216 (1947).

Expository paper, expanding the ideas of earlier ones [Dedebant and Wehrlé, Portugaliae Phys. 1, 95–149 (1944); 179–296 (1945); these Rev. 7, 129, 315]. The present paper is the text of lectures delivered in 1946. *J. L. Doob.*

**Fréchet, Maurice. Positions typiques d'un élément aléatoire de nature quelconque.** Ann. Sci. École Norm. Sup. (3) 65, 211–237 (1948).

The author continues his research on "typical values," that is, generalized means, medians and so on, of random variables with values in a metric space. [See also Rev. Sci. (Rev. Rose Illus.) 82, 483–512 (1944); Ann. Inst. H. Poincaré 10, 215–310 (1948); C. R. Acad. Sci. Paris 226, 1419–1420 (1948); Doss, ibid., 1418–1419 (1948); these Rev. 8, 141; 10, 311; 9, 520.] He gives several definitions, of which the following is an example. Let $U$ be a random variable. Then if $\delta$ is a point of the range space of $U$, for which dist. $(\delta, \lambda)\leqq$median dist. $(U, \lambda)$ for all $\lambda$ in this range space, $\delta$ is a generalized median. *J. L. Doob* (Urbana, Ill.).

**Fréchet, Maurice. On two new chapters in the theory of probability.** Math. Mag. 22, 1–12 (1948).

Chapter I: probabilities associated with any system of events. Chapter II: abstract random variables (measurable functions with values in a certain range space, defined on an abstract measure space; the author assumes that the range space is a metric space). The generalizations of the, standard concepts of dispersion, convergence and so on are discussed. See also the author's earlier paper [Revue Sci. (Rev. Rose Illus.) 82, 483–512 (1944); these Rev. 8, 141]. *J. L. Doob* (Urbana, Ill.).

**Fréchet, Maurice. Les espaces abstraits et leur utilité en statistique théorique et même en statistique appliquée.** J. Soc. Statist. Paris 88, 410–421 (1947).

This report on the author's studies of nonnumerically valued chance variables is similar to his recent expository paper [see the preceding review]. It is assumed that the abstract random elements form a metric space. Definitions and properties used in the theory of random numbers may then be extended to the theory of abstract random elements. The author confines himself to the study of stochastic convergence, typical positions and measures of dispersion [cf. Ann. Univ. Lyon. Sect. A. (3) 9, 5–26 (1946); these Rev. 8, 472]. Finally he mentions possible applications such as the study of cranial sections. *E. Lukacs.*

## *Mathematical Statistics*

**Risser, R. Essai sur les courbes de distribution statistique.** Assoc. Actuair. Belges. Bull. no. 54, 41–72 (1948).

K. Pearson derived the differential equation of his system of curves from the limiting form of the difference equation $[f(x+1)-f(x)]\div\frac{1}{2}[f(x+1)+f(x)]$, the probability function $f(x)$ being either the Bernoulli or the hypergeometric. The author studies the differential equation $y/y'=N/D$, where $N$ is a polynomial of degree three or less and $D$ is of degree four in $x$, based on the difference equations of the form $[f(x+1)-f(x)]\div f(x)$ or $[f(x+1)-f(x-1)]\div f(x)$. Many cases are classified if $N$ is of degree one or a constant. Methods of fitting such curves by moments and a brief discussion of the fourth degree exponential distribution function conclude the article. *L. A. Aroian.*

**Aroian, Leo A. The fourth degree exponential distribution function.** Ann. Math. Statistics 19, 589–592 (1948).

Given the exponential distribution function

$$f(t)=k\exp\{-\sum_{j=1}^{4}\beta_j t^j\},$$

where $t=(x-m)/\sigma$ and $r_1\leqq t\leqq r_2$, the method of moments is used to obtain preliminary estimates of the $\{\beta_i\}$ in terms of the first six moments of $t$. These estimates can be improved by iteration in the maximum likelihood estimating equations, which are also presented in the article.

*R. L. Anderson* (Raleigh, N. C.).

**Choudhary, Nazir Ahmad. A generalization of binomial, Lexian and Poisson distributions.** Math. Student 15 (1947), 8 (1948).

The author finds the moment generating function, the first four cumulants, and central moments of the probability function associated with

$$f=(p_1+q_1)^\alpha(p_2+q_2)^\beta\cdots(p_N+q_N)^\eta,$$

$\alpha, \beta, \eta$ positive integers or zero, $\alpha+\beta+\cdots+\eta=N$, including as special cases the binomial and Lexis distributions.

*L. A. Aroian* (New York, N. Y.).

**Hartley, H. O. Approximation errors in distributions of independent variates.** Biometrika 35, 417–418 (1948).

Let $\varphi(x, y)$ be a real measurable function of $x$ and $y$, defined in the entire plane and monotonically increasing in both $x$ and $y$. Let $c$ be any real number and $Q$ the region $\{\varphi(x, y)<c\}$. Let $F(x)$, $G(x)$, $f(x)$, $g(x)$ be four cumulative distribution functions, and write $m_1=\max_x|F(x)-f(x)|$, $m_2=\max_x|G(x)-g(x)|$. It follows that

$$\iint_Q dF(x)dG(y)-\iint_Q df(x)dg(y)$$

does not exceed $m_1+m_2$ in absolute value. The author proves a special case of this. *J. Wolfowitz* (New York, N. Y.).

**Leslie, P. H. Some further notes on the use of matrices in population mathematics.** Biometrika 35, 213–245 (1948).

The author elaborates on his earlier paper [Biometrika 33, 183–212 (1945); these Rev. 7, 465] by discussing some characteristics of population growth. In a footnote he mentions that E. G. Lewis [Sankhyā 6, 93–96 (1942)] has investigated the problem discussed in the author's previous paper. *E. Lukacs* (China Lake, Calif.).

**Daniels, H. E. A property of rank correlations.** Biometrika 35, 416–417 (1948).

The author introduced a new measure of rank correlation, $\Gamma=\sum a_{ij}b_{ij}/\{\sum a_{ij}^2\sum b_{ij}^2\}^{\frac{1}{2}}$, where $a_{ij}$ and $b_{ij}$ are scores for corresponding pairs of two variables [Biometrika 33, 129–135 (1944); these Rev. 6, 91]. In the present note, he shows that, if any two corresponding pairs of ranks do not agree

in order, the value of $\Gamma$ increases when the members of one of the pairs are interchanged provided the scores do not decrease with increasing rank separation and are not zero except for tied ranks. *R. L. Anderson.*

**Quenouille, M. H. Some results in the testing of serial correlation coefficients.** Biometrika 35, 261–267 (1948).

Let $r$ denote the serial correlation coefficient

$$\sum_{i=1}^{n-l} x_i x_{i+l} \Big/ \left\{ \sum_{i=1}^{n-l} x_i^2 \sum_{i=1}^{n-l} x_{i+l}^2 \right\}^{\frac{1}{2}},$$

for which the reviewer has derived a distribution for the circular approximation when the true correlation $\rho=0$ [Ann. Math. Statistics 13, 1–13 (1942); these Rev. 4, 22]. The present article presents various sets of values of $r$ for samples drawn from rectangular and binomial distributions with $|\rho|$ ranging from 0 to 1. Some actual pressure and price data were also studied. These data have been used to demonstrate that for $\rho=0$ the distribution of $r$ could be approximated by that of the ordinary correlation coefficient as derived by Rubin [Ann. Math. Statistics 16, 211–215 (1945); these Rev. 7, 20] and for $\rho\neq 0$ by a similar approximation as derived by Madow [Ann. Math. Statistics 16, 308–310 (1945); these Rev. 7, 131]. It is also shown that the normal approximation to the distribution of $r$ is adequate for as few as 20 degrees of freedom and probably for even less. Finally an investigation was made of the correlation between two serially correlated series.

*R. L. Anderson* (Raleigh, N. C.).

**Godwin, H. J. A further note on the mean deviation.** Biometrika 35, 304–309 (1948).

A geometrical argument is used to obtain the distribution of the mean absolute deviation from the sample mean for samples from a normal population. This distribution was found earlier by another method [Biometrika 33, 254–256 (1945); these Rev. 8, 42]. Various methods of approximating the distribution are investigated and one approximation found which gives percentage points to three figures when the sample size exceeds 35. *A. M. Mood.*

**Krishna Sastry, K. V. On a Bessel function of the second kind and Wilks' $Z$-distribution.** Proc. Indian Acad. Sci., Sect. A. 28, 532–536 (1948).

Let $Z=B\theta_1\theta_2$, where $\theta_i$ has the probability function $\{1/\Gamma(a_i)\}\theta_i^{a_i-1}e^{-\theta_i}$, $i=1, 2$. C. T. Hsu [Ann. Math. Statistics 12, 279–292 (1941); these Rev. 3, 174] has obtained the distribution of $Z$ if $B=1$, $a_1=a_2=(N-1)/2$, in a form involving a quadrature. The author reduces Hsu's result to (*) $cx^m k_n(x)$, where $k_n(x)$ is a Bessel function of the second kind. The distribution of the quotient $v=x_1/x_2$ is obtained if $x_1$ and $x_2$ are distributed according to (*) with parameters $m_1$, $n_1$, and $m_2$, $n_2$. As a special case of $v$ the distribution of $Z_1/Z_2$ (a problem posed by Hsu) is solved in terms of the hypergeometric function when $Z_1$ and $Z_2$ are distributed as $Z$.

*L. A. Aroian* (New York, N. Y.).

**Narain, R. D. A new approach to sampling distributions of the multivariate normal theory. I.** J. Indian Soc. Agric. Statistics 1, 59–69 (1948).

For $N$ independent observations on $p$ variables $x_1, \cdots, x_p$ which have a joint normal distribution, the author defines $b_{r,r-u}$ as the sample partial correlation coefficient of $x_r$ on $x_{r-u}$ keeping $x_1, \cdots, x_{r-u-1}$ constant, and $\chi_r^2$ as the sum of squares of the residuals in the regression of $x_r$ on $x_1, \cdots, x_{r-1}$. The conditional distribution of $\chi_r^2, b_{r,1}, b_{r,2}, \cdots, b_{r,r-1}$ subject to constant values of $\chi_u^2, b_{u,v}, u=1, \cdots, r-1; v=1, \cdots, u-1$ is obtained analytically. From this and the joint distribution of all the $\chi_r^2$ and $b_{r,r-u}$ which easily follows, various distributions of interest may be derived. Among them is the Wishart distribution of the sample product moments and the distribution of the multiple correlation coefficient.

*H. Chernoff* (Chicago, Ill.).

***Wolfowitz, J. Non-parametric statistical inference.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 93–113. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

The paper is mainly expository, summarizing some recent developments in the theory of nonparametric inference. A new result of the author is a formula for the asymptotic variance of $U$, the number of runs in a set of observations [cf. Wald and the author, Ann. Math. Statistics 11, 147–162 (1940); these Rev. 1, 348], when the observations are drawn from two different distributions. *D. Blackwell.*

***Hsu, P. L. The limiting distribution of functions of sample means and application to testing hypotheses.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 359–402. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

The author generalizes a theorem of the reviewer [Ann. Math. Statistics 6, 160–169 (1935)] on the large sample distributions of functions of sample means and then applies it to find the large sample distribution of a large number of functions used in testing various hypotheses. For example, he considers a 4-variate distribution and finds a test of the hypothesis that the three tetrad differences vanish. If the hypothesis is true the test function has a $\chi^2$ distribution with 2 degrees of freedom. *J. L. Doob* (Urbana, Ill.).

**Votaw, David F., Jr. Testing compound symmetry in a normal multivariate distribution.** Ann. Math. Statistics 19, 447–473 (1948).

Let $X_i$ ($i=1, \cdots, t$) have a joint normal distribution. For a partition of $t$ so that $t=b+n_1+\cdots+n_h$, let the set of $t$ variates $X_i$ be grouped into $b+h$ sets with 1 variate in each of the first $b$ sets and $n_i$ in the $(b+i)$th set. The first hypothesis considered specifies that within each set of variates the means are equal, the variances are equal, and the covariances are equal and that between any two distinct sets of variates the covariances are equal. Five other hypotheses of "compound symmetry" of one distribution are defined. For testing each of these hypotheses on the basis of a sample of $N$ observations the likelihood ratio criterion is obtained. The author gives the moments of each criterion, an exact distribution as the distribution of a product of independent beta variates, and the limiting distribution for $N\to\infty$. Some special distributions and examples are given. When one of the above hypotheses holds for each of $k$ distributions one can test a related hypothesis of equality of certain parameters in all $k$ distributions on the basis of $k$ samples, one from each distribution. These six corresponding $k$-sample problems are treated in the same fashion as the one-sample problems. [The author infers that certain criteria are independent for the inadequate reason that, if $X$ and $Y$ $(0\leqq X, Y\leqq 1)$ have a joint distribution, the equality $E(XY)^h=EX^hEY^h$ $(h=1, 2, \cdots)$ implies independence of $X$ and $Y$ (counter-examples can be given).]

*T. W. Anderson* (New York, N. Y.).

**Huzurbazar, V. S. The likelihood equation, consistency and the maxima of the likelihood function.** Ann. Eugenics 14, 185–200 (1948).

Under the assumptions of Cramér [Mathematical Methods of Statistics, Princeton University Press, 1946, p. 500; these Rev. 8, 39] the author proves the following theorems. (a) With a probability approaching one, the logarithm of the likelihood has a local maximum at a consistent solution of the likelihood equation. (b) A consistent solution of the likelihood equation is unique. [Consistency is a property of a sequence of estimates. The reviewer's understanding of this statement is as follows. The assumption that, with probability approaching one, the likelihood equation will have two roots within any fixed but arbitrary interval of the parameter line which contains the true parameter point in its interior, leads to a contradiction.]

The author then discusses densities of the form $\exp\{u_1(\theta)\cdot u_2(x)+u_3(x)+u_4(\theta)\}$ and gives necessary and sufficient conditions for the existence, uniqueness, and consistency of the solution of the likelihood equation. These conditions are not probabilistic in character.

The author concludes with a discussion of the case when the range of the distribution is a function of $\theta$. This does not lend itself readily to a summary. *J. Wolfowitz.*

**Jones, A. E. Systematic sampling of continuous parameter populations.** Biometrika 35, 283–290 (1948).

Let $T_0\leqq t\leqq T$, and $z(t)=c+X(t)+\epsilon(t)$, where $c$ is a constant; $\epsilon(t)$, called the error of observation, is a stochastic process such that $E\epsilon(t)=0$, $E\epsilon(t_1)\epsilon(t_2)=0$ for $t_1\neq t_2$, and $X(t)$ is a stationary (wide sense) stochastic process with zero mean, variance $\lambda$, correlation function $e^{-q|t|}$, and such that $E\epsilon(t_1)X(t_2)=0$ for all $t_1$ and $t_2$. It is assumed that $\lambda$ and $q$ are known; $c$ may be known or not, and the theory applies also to the case where $c$ is a linear function of $t$. The problem is the following: to find $n$ fixed points $t_1<\cdots<t_n$, and $n$ weights $v_1, \cdots, v_n$, such that $\sum v_i z(t_i)$ is a best estimate of $I=\int_{T_0}^T X(t)dt$ in the least squares sense, i.e., that $E[I-\sum v_i z(t_i)]^2$ is a minimum. [It is to be noted that what is being estimated here is itself a random variable.] The author proves that this occurs when $v_1=\cdots=v_n$, $t_2-t_1=t_3-t_2=\cdots=t_n-t_{n-1}=$

$$q^{-1}\log\left[\left\{k+\frac{(\lambda-\mu)}{2\lambda}\,\frac{(T-T_0)}{n}q\right\}\left\{k-\frac{(\lambda-\mu)}{2\lambda}\,\frac{(T-T_0)}{n}q\right\}^{-1}\right],$$

where $k$ is a root of

$$\left\{k-\frac{(\lambda+\mu)}{2\lambda}\,\frac{(T-T_0)}{n}q\right\}^{n+1}\times\left\{k+\frac{(\lambda-\mu)}{2\lambda}\,\frac{(T-T_0)}{n}q\right\}^{1-n}=e^{-q(T-T_0)},$$

and

$$t_1-T_0=T-t_n=q^{-1}\log\left\{k-\frac{(\lambda+\mu)(T-T_0)q}{2\lambda n}\right\}.$$

The author also proves that then $h=(t_1-T_0)/(t_2-t_1)$ does not depend on $\lambda$ and $\mu$ and obtains approximate values for $h$. *J. Wolfowitz* (New York, N. Y.).

**Kendall, M. G. Continuation of Dr. Jones's paper.** Biometrika 35, 291–296 (1948).

Detailed comment on Jones's results reviewed above. The comments do not lend themselves to ready summary. *J. Wolfowitz* (New York, N. Y.).

**✱Neyman, J. Contribution to the theory of the $\chi^2$ test.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 239–273. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

It is well known that maximum likelihood (ML) estimates have, under general conditions, the following properties: (a) consistency, (b) asymptotic normality, (c) minimal variance of the limiting distribution. The author states that these three properties form "the only rational basis for preferring" ML estimates to others. Consequently an estimate which possesses these properties will, "as far as its asymptotic properties are concerned, be just as good an estimate" as the ML estimate. In particular, an estimate which is an ML estimate plus a chance variable which converges to zero stochastically possesses properties (a), (b), and (c).

The author considers especially the following problem. The probabilities in several fixed multinomial distributions are known functions, with certain regularity properties, of a number of unknown parameters, and the parameters are to be estimated from the sample proportions. The ML estimates then have the property (d) that they possess continuous partial derivatives with respect to the sample proportions. Estimates which have properties (a), (b), (c), (d) are called by the author "best asymptotically normal" (BAN). The author studies BAN estimates "with the hope that some of them can be more easily computed than the ML estimates." [A related idea, though not developed so explicitly and in such detail as in the present paper, is to be found in Cramér, Mathematical Methods of Statistics, Princeton University Press, 1946, sections 30.3, 33.4; these Rev. 8, 39. The present paper was actually submitted for publication in 1945.] Various theorems, which cannot readily be summarized, give necessary and sufficient conditions for estimates to be BAN. Methods for obtaining BAN estimates by solving linear equations are given. One theorem gives two types of the $\chi^2$ test which are consistent and in the limit equivalent to the likelihood ratio test. The application of these tests requires the computation of BAN estimates. *J. Wolfowitz* (New York, N. Y.).

**✱Lehmann, E. L. Some comments on large sample tests.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 451–457. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

Two examples are constructed which show that tests which have the property of being asymptotically most powerful do not necessarily have much to recommend them. A sequence of regions $\{W_n\}$ is an asymptotically most powerful test if for any other test $\{Z_n\}$ with the same significance level there exists an $N(\epsilon)$, given $\epsilon$, such that $n\geqq N$ implies $P(W_n|\theta)>P(Z_n|\theta)-\epsilon$ for all $\theta$, where $P(W_n|\theta)$ is the power function for tests of $\theta=\theta_0$ based on $W_n$. One example gives an asymptotically most powerful test $W_n$ and another test $Z_n$ such that for all $\theta$ one has $[1-P(Z_n|\theta)]/[1-P(W_n|\theta)]\rightarrow 0$ as $n\rightarrow\infty$. The second example shows that given any sequence of numbers $r_n\rightarrow 1$, there exists an asymptotically most powerful test $W_n$ such that $P(W_n|\theta)\leqq r_n$ for all $\theta$. *A. M. Mood.*

**Pearson, E. S., and Merrington, Maxine. $2\times 2$ tables; the power function of the test on a randomized experiment.** Biometrika 35, 331–345 (1948).

By approximate methods the authors obtain and tabulate the power of the $\chi^2$ test for the double dichotomy

described in the following table (reproduced from table 3a of the paper):

| | React if given A or B | React if given B | React to neither treatment | Total |
|---|---|---|---|---|
| Treatment A | $x_1$ | $y_1$ | $z_1$ | $m$ |
| Treatment B | $x_2$ | $y_2$ | $z_2$ | $n$ |
| Total | $X$ | $Y$ | $Z$ | $N$ |

The observed quantities are $x_1, y_1+z_1, x_2+y_2, z_2$. The null hypothesis is that $Y=0$. The authors take $m=n$.

*J. Wolfowitz* (New York, N. Y.).

**Ghosh, M. N. A test for field uniformity based on the space correlation method.** Sankhyā 9, 39–46 (1948).

The space correlation coefficient is the two-dimensional analogue of the serial correlation coefficient. It is shown that the nonparametric moments of the covariance

$$k^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij}-k^{-1}\sum\sum x_{ij})(x_{i+m',j+n'}-k^{-1}\sum\sum x_{i+m',j+n'}),$$

$k=mn$, converge stochastically to the moments of a normal distribution with zero mean. The exact first and second moments of the covariance have been calculated. Hence it is possible to test for the randomness of large samples from a two-dimensional field by use of normal probability tables.

*R. L. Anderson* (Raleigh, N. C.).

# TOPOLOGY

**★Kuratowski, Casimir. Topologie. I. Espaces Métrisables, Espaces Complets.** 2d ed. Monografie Matematyczne, vol. 20. Warszawa-Wrocław, 1948. xi+450 pp. $7.50.

This is a revised and enlarged edition of the first edition [1933]. A few changes of the text of the original edition seem to have been dictated by the desire for a more perfect logical order. However, the major change is the addition of about 165 pages of new material, motivated by one of the following aims: (1) to give a definitive treatment of topics considered in the first edition, using results discovered since 1933, and (2) to fit the first volume better to the second volume, which is to appear soon. There has been no essential modification in arrangement, so that in the main a reader of volume II will be able to locate references in the old volume I without difficulty. Detailed remarks below are limited to the new material.

The introduction contains additional theorems on operations in set theory, with a discussion of Lusin's sieve and its relation to operation ($\mathfrak{A}$). Chapter I, on topological spaces, is as in the old edition. Chapter II, on separable metric spaces, contains several changes. In the discussion of $\mathfrak{L}^*$ spaces (spaces with convergent sequences) there is new material on compact $\mathfrak{L}^*$ spaces, including the function space $\mathfrak{Y}^{\mathfrak{X}}$, where $\mathfrak{X}$ and $\mathfrak{Y}$ are $\mathfrak{L}^*$ spaces. There is a longer discussion of function spaces, including theorems on extending continuous functions, and in particular, Tietze's extension theorem. The most important addition is an entire section [§ 23] on simplexes, complexes and polytopes. It contains the fundamental theorems on the dimension and the fixed point property of the $n$-simplex, and in addition a series of theorems on the approximation of arbitrary continuous transformations by $\chi$-transformations. (A $\chi$-transformation is the familiar continuous transformation associated with an open covering $\alpha$ of a space $\mathfrak{X}$, and mapping $\mathfrak{X}$ into a geometric realization of the nerve of $\alpha$.)

The principal additions to chapter III are concerned with theorems on projective sets and singular spaces. In particular, there are several theorems on Borel sets and the characterization of the Borel class. There are theorems on the invariance of the projective class under operation ($\mathfrak{A}$) and Lusin's sieve, and as an application of these theorems there is a proof that the Lebesgue set is a projective set. The section on singular spaces has been augmented by a fuller discussion of $\lambda$-spaces (definition: every countable subset is a $G_\delta$), additional theorems on $\sigma$-spaces (definition: every $F_\sigma$ is a $G_\delta$), and two new paragraphs on $\nu$-spaces (definition: every nowhere dense subset is countable). Much of this material has been developed since the appearance of the first edition.

This book is excellent for reference, and as a text, especially for the student who has had some introduction to topology. The author has demonstrated great ability in organizing a great mass of material in strict logical order and thus avoiding repetitions. This very virtue, however, results in a book which may seem too difficult to the beginning student who tackles it without the aid of an instructor.

*J. H. Roberts* (Durham, N. C.).

**Koseki, Ken-iti. Über die Begrenzung eines besonderen Gebietes.** Jap. J. Math. 19, 285–299 (1947).

Theorems on the structure of sets $B$ in the plane which are the common boundary of two connected open sets, $U$ and $V$, are proved, closely related to work of Whyburn, Janiszewski, and others. A typical result is that if $B$ is bounded, then no continuum of condensation of $B$ or indecomposable subcontinuum of $B$ contains more than two points accessible from all sides from both $U$ and $V$.

*G. S. Young* (Ann Arbor, Mich.).

**Aleksandrov, P. S. On the concept of space in topology.** Uspehi Matem. Nauk (N.S.) 2, no. 1(17), 5–57 (1947). (Russian)

This is an exposition of the theory of regular topological spaces and their extensions and bicompactifications. It presents the principal work in this field of Urysohn, the author, Tychonoff, Čech, Kurosh, and Wallman; it is self-contained and very clear. Section 1 begins with Hausdorff spaces and a variety of separation properties, with emphasis on complete regularity. Section 2 gives the equivalent formulations of bicompactness; section 3 is devoted to the product of bicompacta and the imbedding in such products of given completely regular spaces. Section 4 deals with the maximal bicompact extension of completely regular spaces and other related extensions of regular spaces. Section 5 treats bicompacta as images of the dyadic dicontinuum $D_\tau$ and introduces an analogous set $F_\tau$. This is also the product of $\tau$ sets, each a pair of points. However, the pair is now chosen as a $T_0$-space, called a connected doublet, whose open sets are three: the null set, the pair of points, and one point of the pair. In section 6 the projection-spectrum is defined whose members are finite discrete spaces related by mappings. The barycentric subdivision of simplexes is also treated here, the totality of successive subdivisions being interpreted as a spectrum. It is shown how

bicompacta may be realized naturally as the limit spaces of spectra. In section 7 it is proved that the lower limit of the spectrum associated with an arbitrary normal space is the maximal bicompactification of that space.

*L. Zippin* (Flushing, N. Y.).

**Ramanathan, A. Minimal-bicompact spaces.** J. Indian Math. Soc. (N.S.) 12, 40–46 (1948). compact

Having recently considered noncompact maximal Hausdorff spaces [same J. (N.S.) 11, 73–80 (1947); these Rev. 10, 137] the author now turns to minimal ~~Hausdorff~~ spaces and produces a non-Hausdorff example [Hing Tong, in an abstract, Bull. Amer. Math. Soc. 54, 478–479 (1948), has also produced such a space]. The present example is finally used to show that a point having no countable basis can nevertheless be a $G_\delta$ and the limit of a sequence.

*R. Arens* (Los Angeles, Calif.).

**Balachandran, V. K. Minimal bicompact space.** J. Indian Math. Soc. (N.S.) 12, 47–48 (1948).

Another example of a non-Hausdorff minimal-compact space is given, simpler than Ramanathan's or Hing Tong's [cf. the preceding review]. We reproduce it here. Let $x=\{x, y, x_{ij}\}$, where $i, j=1, 2, \cdots$. Let each $x_{ij}$ be isolated; let neighborhoods of $x$ be $X$ minus finitely many elements from each row, let neighborhoods of $y$ be $X$ minus finitely many rows. *R. Arens* (Los Angeles, Calif..)

**Beda Neto, L. Contribution to the study of the theory of functions.** Revista Fac. Ci. Univ. Coimbra 17, 42–86 (1948). (Portuguese)

Continuation of a memoir begun in 1934. Cf. these Rev. 8, 285.

**Abdelhay, J. Characterization of regular and normal topological spaces by means of coverings.** Gaz. Mat., Lisboa 9, no. 37–38, 8–9 (1948). (Portuguese)

The author remarks that the known proof of the regularity and normality of paracompact spaces can be used to give a characterization of these two properties.

*L. Nachbin* (Chicago, Ill.).

**Mendonça de Albuquerque, Luís. On the first class of set functions.** Revista Fac. Ci. Univ. Coimbra 17, 111–122 (1948). (Portuguese)

**Mendonça de Albuquerque, Luís. Characterization of the abstract spaces by the family of isolated sets.** Revista Fac. Ci. Univ. Coimbra 17, 150–156 (1948). (Portuguese)

The author indicates some characterizations of the notion of topological space. It is the reviewer's impression, however, that not all of his statements are correct.

*L. Nachbin* (Chicago, Ill.).

**Zaremba, S. K. Contribution à la théorie des réseaux sur les surfaces fermées.** Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Sér. A. Sci. Math. 1939, 133–149 (1946).

Let $F$ be a closed orientable surface. A set $R$ of curves on $F$ will be said to form a net regular on $F$ if to any point $M$ whatever of $F$ (except for a finite number of singular points) one may make correspond a neighborhood $U(M)$ of $M$ such that the simple continuous arcs, except perhaps their extremities, in $U(M)$ and belonging to the curves of $R$, may be separated into two families, say $A$ and $B$, such that there exists a homeomorphism between $U(M)$ and the Euclidean plane transforming $A$ and $B$ into two families of segments of lines parallel, respectively, to two distinct directions, each of these two families being composed of all the segments parallel to the respective directions except perhaps their extremities, in the homeomorphism of $U(M)$. H. Hamburger [Math. Z. 19, 50–66 (1924)] and B. de Kerékjártó [Vorlesungen über Topologie, v. 1, Springer, Berlin, 1923] in their studies of the topic based their considerations on two different kinds of indices of singular points. The present paper defines an index (combinatorial index) of certain curvilinear normal polygons as follows. A polygon of $R$ is called normal if it is closed and simple, its sides being arcs of the curves on $R$, two consecutive sides being always in a common neighborhood $U(M)$ of their vertex $M$. At any vertex $M$ two consecutive sides either belong to the same family (in which case $M$ is said to be neutral) or belong to the families $A$ and $B$, their arcs (except possibly their extremities) belonging to $U(M)$. In this latter case, supposing the orientation of $F$ positive, and the normal polygon of $R$ oriented, $M$ is called a positive or negative vertex according to the following scheme. The two sides through $M$ are extended to the boundary of $U(M)$, $LM$ and $MN$ being the extensions, and oriented in conformity to that of the normal polygon. The curvilinear triangle $LMN$ is formed from the sides of the polygon through $M$ and that one of the arcs determined on $U(M)$ not containing the extensions of the arcs through $M$ in the contrary sense. That portion of the interior of $U(M)$ also interior to $LMN$ being considered oriented positively (that is, the same as $F$), if the oriented triangle $LMN$ forms its boundary, then $M$ is said to be positive, but if $LMN$ is not the oriented boundary of that portion of $U(M)$ then $M$ is said to be negative. A curvilinear oriented normal polygon $a$ is said to have the index ind $(a)=1-\frac{1}{4}s+\frac{1}{4}r$, where $s$ is the number of positive and $r$ the number of negative vertices of $a$.

Some typical theorems may be stated as follows. Let $a$ and $b$ be two normal oriented polygons whose common part forms an open polygon (considered with opposite orientations) such that the algebraic sum $a+b$ is again a normal oriented polygon; then ind $(a+b)-[\text{ind }(a)+\text{ind }(b)]$ is an integer. In particular, if one at least of $a$, $b$ forms the boundary of a domain oriented positively and if the other is disjoint with the interior of the first, then ind $(a+b)=$ ind $(a)+$ ind $(b)$; but if the second of these polygons is contained in the domain mentioned above then ind $(a+b)=$ ind $(a)+$ ind $(b)-2$. Again, if $a_1, a_2, \cdots, a_m$ are normal oriented polygons of $R$, and $\lambda_1, \lambda_2, \cdots, \lambda_m$ are arbitrary integers such that $\sum_{i=1}^m \lambda_i a_i=0$, then $\sum_{i=1}^m \lambda_i$ ind $(a_i)$ is an integer. A final theorem may be quoted. Suppose the boundary of a positively oriented portion of $F$ is bounded by $n$ normal oriented polygons of $R$ disjoint by pairs, with respective indices $i_1, i_2, \cdots, i_n$; then the sum $s$ of the indices of the singular points situated on that portion of $F$ (the indices of singular points being defined by Hamburger) is given by the formula $s=\sum_{k=1}^n i_k-2n+2(1-p)$, $p$ being the genus of the surface obtained by contraction of each of the polygons to a point. *V. G. Grove.*

**Haupt, Otto. Bemerkung zu einem Satz von Herrn G. Van der Lyn.** S.-B. Phys.-Med. Soz. Erlangen 71 (1939), 349–352 (1940).

G. van der Lyn [Mathesis 50, 21–24 (1936)] has proved the following. Let $V$ be a topological 3-cell and $\{F\}$ a family of disjoint topological discs whose union is $V$. Further let the boundary of each $F$ be in the boundary of $V$. Then the collection $\{F\}$ forms a continuous decomposition of $V$. Haupt considers the analogous theorem where $V$ is a topo-

logical $n$-cell and each $F$ a topological $(n-1)$-cell. He also assumes that, for each $F$, the part of $F$ in the boundary of $V$ is either just the boundary of $F$ or else is all of $F$. With this additional hypothesis, it is easy to show that the collection $\{F\}$ is simply ordered. But it is also easy to show that a simply ordered decomposition of a compactum is continuous. *E. G. Begle* (New Haven, Conn.).

**Reeb, Georges. Stabilité des feuilles compactes à groupe de Poincaré fini.** C. R. Acad. Sci. Paris **228**, 47–48 (1949).

A previous result of the author [same C. R. **224**, 1613–1614 (1947); these Rev. **8**, 595] is generalized to read as follows. Let $V_n$ be a variety which is leaved (or laminated) in dimension $q$ and let $U$ ($\subset V_n$) be open, simply connected, verifying conditions $C_1, C_2, C_3$ [see below]. Then there exists a saturated open set $\Omega$, with $\Omega\cap U\neq 0$, all of whose complete leaves or laminae are compact. The conditions $C_1, C_2, C_3$ concern the structure of the covering of $\bar{U}$ by the open sets, occurring in the "atlas" which defines the laminated structure of $V_n$. (The atlas consists of homeomorphisms of these open sets into Euclidean $n$-space, with the components of the inverse images of the plane varieties $x_r=$constant ($r=q+1, \cdots, n$) defining the elementary laminae or "plaques.") "Stability" refers to the fact that the conditions remain verified if one changes the laminated structure of $V_n$ slightly. *H. Samelson* (Ann Arbor, Mich.).

**Borsuk, Karol. On the imbedding of systems of compacta in simplicial complexes.** Fund. Math. **35**, 217–234 (1948).

Let $A_1, \cdots, A_n$ be a finite covering of a finite dimensional compact metric space $X$ by closed sets. It is proved that $X$ can be imbedded in a simplicial complex $K$ covered by simplexes $\Delta_1, \cdots, \Delta_n$ such that $A_i=X\cap\Delta_i$ for $i=1, \cdots, n$ and such that the nerve of the decomposition of $K$ is the same as that of $X$. If the sets $A_i$ and all their finite intersections are absolute retracts then $X$ is a deformation retract of $K$. *S. Eilenberg* (New York, N. Y.).

**Glezerman, M., and Pontryagin, L. Intersections in manifolds.** Uspehi Matem. Nauk (N.S.) **2**, no. 1(17), 58–155 (1947). (Russian)

This is a text based on a course of lectures delivered by Pontryagin, compiled under his direction several years ago, and recently somewhat revised. Its primary purpose is a systematic and unexceptionable exposition of the "intersection-ring" of a complex. This is the algebraic ring, designated the Lefschetz ring, based upon the direct sum of the Betti groups of the given complex in which the product of two cycles is taken to be the oriented cycle which is their intersection in the sense due to Lefschetz [Trans. Amer. Math. Soc. **28**, 1–49 (1926); there is no reference to the same author's "Algebraic Topology," Amer. Math. Soc. Colloquium Publ., v. 27, New York, 1942; these Rev. **4**, 84]. The extensive generalizations of this theory due to Alexander and Kolmogoroff are mentioned, and it is proved in the concluding section of the paper that the Lefschetz ring is isomorphic to the Alexander ring [Ann. of Math. (2) **37**, 698–708 (1936)]. This fact is given as proof that the Lefschetz ring is a topological invariant. To demonstrate the isomorphism the authors finally restrict the theory to the class of homology-manifolds and make use of the Poincaré duality expressed in terms of cohomology theory.

The emphasis of the presentation is less upon the generality of the theory than upon what the authors call the "thoroughly pedantic" analysis of its details. They serve notice that the details are numerous and the thorough analysis formidable. These details have to do with three main points. First the notion of "general position," second the orientation of the intersection of chains, third the existence within homology classes of representatives in general position. Forty pages comprising the first two sections are devoted to the notion of "general position" and the orientation of polyhedra contained in a Euclidean space. Section three discusses complexes and a class of chains of the complex called polygonal chains which are in sufficiently general position in the complex (with respect to its cells of highest dimension). The fourth section deals with barycentric subdivisions showing incidentally that the Lefschetz ring is invariant under subdivision. The main purpose of the section is to provide a method of subdivision which will lead to the isomorphism of the Lefschetz and Alexander rings. This isomorphism is established in the fifth and final section. *L. Zippin* (Flushing, N. Y.).

**Mayer, W. Duality theorems.** Fund. Math. **35**, 188–202 (1948).

Proofs are given of the Poincaré and Alexander duality theorems, the latter in a generalized form which gives algebraic relations which obtain between the homology groups of any orientable manifold and those of a closed sub-polytope. The author uses extensively the algebraic theory previously developed by him [Ann. of Math. (2) **46**, 1–28, 29–57 (1945); these Rev. **6**, 280] and the major part of the paper is devoted to proving various results of a geometric nature which enable him to apply his earlier theory. The basic linking used in the proofs is defined via the Whitney cap product with the fundamental cycle of the manifold. [The machinery of exact homomorphism sequences, together with algebraic devices similar to those of the author, have been employed by E. Pitcher and the reviewer to establish a theorem which includes the Poincaré and Alexander-Pontrjagin theorems, Ann. of Math. (2) **48**, 682–709 (1947); these Rev. **9**, 52.] *J. L. Kelley.*

**Čogošvili, G. S. On duality relations in topological spaces.** Uspehi Matem. Nauk (N.S.) **1**, no. 5-6(15-16), 247–250 (1946). (Russian)

This is a descriptive account of the author's doctoral dissertation. The substance of results here stated in a somewhat general way are covered in a note of the same title [C. R. (Doklady) Acad. Sci. URSS **46**, 131–132 (1945); these Rev. **7**, 37]. *L. Zippin* (Flushing, N. Y.).

**Morse, Marston. A positive, lower semi-continuous, nondegenerate function on a metric space.** Fund. Math. **35**, 47–78 (1948).

Let $F$ be a function with properties indicated in the title defined on a metric space $X$. The pair $(X, F)$ is subjected to several axioms. These axioms assert the existence of set $(\sigma)$ (the critical points of $F$), its distribution in $X$, the existence of certain homotopies in the neighborhood of each point of $(\sigma)$, and the existence of certain homotopies in some larger sets. It then turns out that $(\sigma)$ is the set of all "homotopically critical points" in the sense of the author [Functional Topology and Abstract Variational Theory, Mémor. Sci. Math., no. 92, Gauthier-Villars, Paris, 1939]. Using these axioms and singular homology theory, the usual Morse inequalities between the Betti numbers of $X$ and the type numbers of the points of $(\sigma)$ are established. It is sketched how these axioms can be proved, using existing

techniques in the calculus of variations in the large, in the case where $X$ is the space of all curves joining two fixed points on a manifold of class $C^{(4)}$ and $F$ is a line integral $\int f(x, \dot{x})dt$, where the function $f(x, \dot{x})$ is suitably restricted.

*S. Eilenberg* (New York, N. Y.).

**Èl'sgol'c, L. È. The variation of the topological structure of level surfaces.** Mat. Sbornik N.S. **23**(65), 399–418 (1948). (Russian)

This is a classification of the possible changes in Betti numbers, and in the structure of the fundamental group, of the level surfaces $f=c$ of a twice differentiable function $f(p)$ defined on a manifold $M^n$; $p\epsilon M^n$, $c$ real. The author follows the classification of critical points due to M. Morse [Trans. Amer. Math. Soc. **27**, 345–396 (1925)]. It is assumed that the critical points are nondegenerate and correspond to distinct values $f=c$. The author uses the Mayer-Vietoris formula, and a formula due to M. Bockstein [Rec. Math. [Mat. Sbornik] N.S. **9**(51), 365–376 (1941); these Rev. **3**, 60]. The analysis proceeds by cases according to the type of the critical point.

One general theorem, dealing with a so-called "exceptional" case, is as follows: if a manifold $M^{2k}$ has a cycle $z^k$ such that $z^k\cdot z^k$ is not homologous to zero (mod 2), then there exists at least one critical point $p$ such that the Betti numbers (mod 2) of the level surfaces do not change as $x$ passes through the value $c=f(p)$; and conversely. Some representative cases, dealing with the fundamental group, are as follows (numbered as in the paper): (2) at a critical point of type $k>2$, with $n-k>2$, the fundamental group of the set $(f=x)$ does not change; ($3^a$) at a point of augmenting type $k=1$ and $n-k>2$, the group takes on a new free generator; ($4^a$) for $k=2$, $n-k>2$, the group acquires a new relation on its generators. However, this apparently new relation may be a trivial one.

*L. Zippin* (Flushing, N. Y.).

**Kuratowski, Casimir. Sur l'application de la notion d'homotopie au problème du nombre algébrique des points invariants.** Fund. Math. **34**, 261–271 (1947).

Consider a bounded closed subset $E$ of the Euclidean plane and a continuous mapping $f$ of $E$ into itself which has no fixed points on the boundary $F$ of $E$. There exist [Kuratowski, Fund. Math. **33**, 316–367 (1945); these Rev. **8**, 50] a continuous function $u$ and a rational function $r$ such that $f(x)-x=e^{u(x)}r(x)$ for every $x\epsilon F$. The excess $v$ of zeros over poles of $r$ (each counted with its multiplicity) depends on $f$ but not on $r$. The author shows that $v=v_f$ is equal to the algebraic number of fixed points (not necessarily isolated) of $f$ in the interior $E-F$ of $E$. Using this result and the Bruschlinsky characterization [Math. Ann. **109**, 525–537 (1934)] of the 1-dimensional Betti group, a simple proof of the fixed point formula is given for the case of a compact neighborhood retract of the plane.

*R. H. Fox* (Princeton, N. J.).

**Whitehead, J. H. C. On operators in relative homotopy groups.** Ann. of Math. (2) **49**, 610–640 (1948).

In an arcwise connected space $Q$ let $P$ be the union of $k$ 1-spheres which have in common only a single point $p_0$. Certain classes of deformations over $Q$ of $(P, p_0)$ into $(P, p_0)$ constitute a group $H$ using a naturally defined multiplication. A homomorphism $\theta$ of $H$ into $\pi_1(Q)$ is defined by associating to such a deformation the path described by $p_0$. Consider the standard homomorphism of $\pi_1(P)$ into $A_2(Q, P)$, the group of automorphisms of $\pi_2(Q, P)$. A certain subset $F$ of the related crossed homomorphisms of $\pi_1(P)$ into $\pi_2(Q, P)$ is a group under a certain multiplication which the author defines. There is then introduced in a natural manner a homomorphism $h$ of $F$ into $H$. The sequence $F\xrightarrow{h}H\xrightarrow{\theta}\pi_1(Q)$ is exact. Furthermore there are homomorphisms $\epsilon:\pi_2(Q, P)\rightarrow F$ and $\psi:\pi_1(P)\rightarrow H$ which, together with the identity homomorphism $1:\pi_1(Q)\rightarrow\pi_1(Q)$, define a homomorphism of the exact sequence $\pi_2(Q, P)\xrightarrow{\partial}\pi_1(P)\xrightarrow{\phi}\pi_1(Q)$ into the exact sequence $F\xrightarrow{h}H\xrightarrow{\theta}\pi_1(Q)$. This homomorphism has the additional property that it carries the kernel of $\partial$ into the kernel of $h$. Finally there is a homomorphism $T$ of $H$ into $A_2(Q, P)$ [this seems to be the central idea of the paper] such that $T\psi$ is the standard homomorphism of $\pi_1(P)$ into $A_2(Q, P)$. The group $I_2(Q, P)$ of inner automorphisms of $\pi_2(Q, P)$ is the image under $T$ of the subgroup $h(F)\cap\psi(\pi_1(P))$. Similar results are found for the case that $P$ is the union of $(n-1)$-spheres. If one assumes merely that $P$ is an arcwise connected subset of $Q$ the above concepts may still be considered but the resulting situation is rather complicated.

*R. H. Fox* (Princeton, N. J.).

**Whitehead, George W. On spaces with vanishing low-dimensional homotopy groups.** Proc. Nat. Acad. Sci. U. S. A. **34**, 207–211 (1948).

Der Raum $X$ sei durch stetige Wege zusammenhängend und asphärisch in den Dimensionen $i<n$, mit $n>2$. Ist $f$ eine wesentliche Abbildung der Sphäre $S^{n+1}$ auf die Sphäre $S^n$ (es gibt bis auf homotope nur eine solche Abbildung), $g$ eine Abbildung von $S^n$ in $X$, also $gf$ von $S^{n+1}$ in $X$, so bewirkt die Zuordnung $g\rightarrow gf$ einen Homomorphismus $\eta$ der Homotopiegruppe $\pi_n(X)$ in $\pi_{n+1}(X)$. Es werden einige Zusammenhänge zwischen den Homotopie- und den Homologiegruppen von $X$ und dem Homomorphismus $\eta$ angegeben, nämlich ($H_k$ bezeichne die $k$-te ganzzahlige Homologiegruppe von $X$, $\Sigma_k$ die Untergruppe von $H_k$, deren Elemente stetige Bilder der $S^k$ enthalten, $\pi_k$ die Gruppe $\pi_k(X)$, $\varphi_n\subset\pi_n$ den Kern von $\eta$): (1) $H_{n+1}\cong\pi_{n+1}/\eta\pi_n$; (2) für $n>3$ ist $H_{n+2}/\Sigma_{n+2}\cong\varphi_n/2\pi_n$; (3) für $n>3$ und $\pi_{n+2}=0$ ist $H_{n+3}/\Sigma_{n+3}\cong H_{n+1}/2H_{n+1}+{}_2\pi_n$, wo ${}_2\pi_n$ die aus den Elementen der Ordnung 2 bestehende Untergruppe von $\pi_n$ ist. Für $n>3$ sind also $H_{n+1}$ und $H_{n+2}/\Sigma_{n+2}$ durch $\pi_n$, $\pi_{n+1}$ und $\eta$ bestimmt, und wenn $\pi_{n+2}=0$ ist, auch $H_{n+2}$ und $H_{n+3}/\Sigma_{n+3}$. Ist speziell $\pi_{n+1}=\pi_{n+2}=0$, so folgt aus (1)–(3): $H_{n+1}=0$, $H_{n+2}\cong\pi_n/2\pi_n$ und $H_{n+3}/\Sigma_{n+3}={}_2\pi_n$. Ein Ergebnis von Eilenberg und MacLane [Ann. of Math. (2) **46**, 480–509 (1945); diese Rev. **7**, 137] besagt: Wenn für $i\neq n$, $i<n+k$, alle $\pi_i=0$ sind, so sind die $H_i$, $n\leqq i<n+k$, und $H_{n+k}/\Sigma_{n+k}$ durch $\pi_n$ bestimmt; dieser Satz wird somit hier für $n>3$ und $k=3$ neu bewiesen und überdies präzisiert.

Die Beweise sind nur skizziert. Die wichtigsten Hilfsmittel sind: der Raum $X^*$, der aus $X$ durch Einspannen von Elementen in die Sphärenbilder in $X$, die Erzeugenden von $\pi_n(X)$ angehören, entsteht; die Homologie- und die Homotopiesequenz von $X^*$ mod $X$, die miteinander auf zwei Arten verknüpft werden, nämlich durch die natürliche Abbildung der $\pi_k$ in die $H_k$ (auf die $\Sigma_k$) und durch Verallgemeinerungen des Homomorphismus $\eta$; Angaben über die Struktur der Homotopiegruppen von $X^*$ mod $X$, welche einem unveröffentlichen Ergebnis von J. H. C. Whitehead entnommen werden. Es wird ferner für $n>2$ auf Zusammenhänge mit der Invarianten $k^{n+2}$ von Eilenberg und MacLane

[vgl. für $n=1$: Proc. Nat. Acad. Sci. U. S. A. **32**, 277–280 (1946); diese Rev. **8**, 398] und mit dem von Steenrod eingeführten Homomorphismus $Sq_{n-2}$ [Ann. of Math. (2) **48**, 290–320 (1947); diese Rev. **9**, 154] hingewiesen.
*B. Eckmann* (Zürich).

**Hu, Sze-tsen. On Čech homology groups of retracts.** Fund. Math. **35**, 181–187 (1948).

The author discusses the relative Čech homology groups $H^n(X, A)$ over an arbitrary topological coefficient group $G$, where $X$ is a topological space and $A$ is a closed subset of $X$. The "homology sequence" of groups and homomorphisms involving the groups $H^n(X, A)$, $H^n(X)$ and $H^n(A)$ is introduced. It is asserted that "using an argument of Alexandroff," it can be proved that this sequence is exact (i.e., has the "kernel=image" property) when $G$ is a division closure group. To the best of the reviewer's knowledge this is not the case even when $X$ is compact metric and $G$ is the discrete additive group of rational integers. The sequence is exact when $G$ is compact. The application of the exactness of the homology sequence to the case when $A$ is a retract of $X$ duplicates a similar remark that the author has made in connection with the homotopy sequence [Nederl. Akad. Wetensch., Proc. **50**, 279–287 = Indagationes Math. **9**, 169–177 (1947); these Rev. **8**, 595]. *S. Eilenberg*.

**Hu, Sze-tsen. Extension and classification of the mappings of a finite complex into a topological group of an $n$-sphere.** Ann. of Math. (2) **50**, 158–173 (1949).

The general question of the classification of the homotopy classes of maps $f:X\rightarrow Y$ for two fixed spaces $X$ and $Y$ is discussed in the following two cases: (1) $Y$ is an arcwise connected topological group and $X$ is a cell complex, (2) $Y=S^n$ is the $n$-sphere and $X$ is a cell complex of dimension $m<2n-2$. It is known that in both these cases the homotopy classes can be converted into a group $D$. The main result of the paper is that $D$ has a composition series $D=D^0\supset D^1\supset\cdots\supset D^m=1$ such that $D^{n-1}/D^n$ is isomorphic with the factor group of two subgroups of the cohomology group $H^n(X, \pi_n(Y))$. This classification theorem is preceded by a suitable extension theorem. *S. Eilenberg*.

**Rohlin, V. Homotopy groups.** Uspehi Matem. Nauk (N.S.) 1, no. 5–6(15–16), 175–223 (1946). (Russian)

This is an expository paper on the theory of homotopy groups, formulated for polyhedra, and based upon a completely equivalent but more "geometrical" definition of these groups than the one originally given by Hurewicz. A footnote explains that the paper appears as it was completed in 1940, with supplementary references to some subsequent literature [to 1943]. *L. Zippin* (Flushing, N. Y.).

## GEOMETRY

***Gerretsen, J. C. H. Niet-Euklidische Meetkunde. [Non-Euclidean Geometry.]** 2d ed. J. Noorduijn en Zoon N. V., Gorinchem, 1949. xi+212 pp. (1 plate).

This book provides a synthetic introduction to classical hyperbolic geometry. It begins with a historical chapter which includes Legendre's proof that the angle-sum of a triangle cannot exceed a straight angle (if lines are infinite).

Chapter 2 deals with plane geometry, based on Euclid's first four postulates and the assumption that the angle-sum of a triangle is less than a straight angle. Points at infinity are introduced as an aid to the study of parallelism. A circle is defined as the locus of images of a point by reflection in the lines of a pencil, with the horocycle arising in the limiting case of a pencil of parallels.

Chapter 3 is on solid geometry, beginning with the hyperbolic analogues of the propositions in Euclid XI. The author shows that the intrinsic geometry of the horosphere is Euclidean. He uses Poincaré's conformal model in the Euclidean half-space to prove that the fifth postulate is independent of the other four.

Chapter 4 is a Euclidean interlude, introducing the logarithmic and hyperbolic functions as properties of the ordinary rectangular hyperbola. The development is ingenious, but one might well ask if it is worthwhile to take so much trouble to avoid any mention of the integral calculus.

Chapter 5, on trigonometry, begins with the relation between corresponding arcs of concentric horocycles. After finding the expression for Lobachevsky's angle of parallelism, the author obtains the formulas for a right triangle by comparison with the corresponding triangle of horocycles on a horosphere. Arcs and sectors of circles and horocycles are computed, with some further ingenious devices for avoiding the calculus. Finally, there is a discussion of the possibility that hyperbolic geometry might be applicable to astronomical space provided the absolute unit of length is not less than two million times the diameter of the earth's orbit.

The book is nicely printed, with a good index and bibliography (except that one misses the names of Coolidge and Klein), excellent diagrams, and a fine portrait of Lobachevsky as frontispiece. *H. S. M. Coxeter*.

***Schilling, Friedrich. Die Bewegungstheorie im nichteuklidischen hyperbolischen Raum.** Leibniz-Verlag, München, 1948. Vol. 1, pp. i–vi+1–138; vol. 2, pp. i–iii+139–287. 35 Marks.

In non-Euclidean space, every displacement is expressible as the product of reflections in four planes which can be so chosen that both the first two are perpendicular to both the last two; i.e., the product of "rotations" about two polar lines. When the absolute quadric is real but without real generators, the two polar lines are either two tangents or a secant and an exterior line. Hence a displacement in hyperbolic space is either a parallel displacement or a screw. The former leaves invariant a point on the absolute quadric and a system of concentric horospheres which behave like Euclidean planes. The displacement corresponds in such a plane to the product of two perpendicular translations, which is a single translation. Hence the parallel displacement is simply the product of reflections in two parallel planes. When applied continuously, the locus of any point (without exception) is a horocycle. On the other hand, a screw is the product of a translation along a certain line and a rotation about the same line, so that the locus of a point not on the axis is a helix,

$$x:y:z:w=\sinh \gamma t : c \sin t : c \cos t : \cosh \gamma t,$$

the curve of intersection of the cylinder $y^2+z^2=c^2(w^2-x^2)$ and the ruled helical surface $\tanh^{-1}(x/w)=\gamma \tan^{-1}(y/z)$. The cylinder may be described as a quadric having double contact with the absolute quadric, or as the surface obtained by revolving a hypercycle about its axis.

The author uses homogeneous coordinates $(x, y, z, w)$ with the absolute quadric $x^2+y^2+z^2-w^2=0$, and derives non-

homogeneous coordinates $(x, y, z)$ by setting $w=1$. A Euclidean model is obtained by regarding $x$, $y$, $z$ as Cartesian coordinates in ordinary space. In this model, the hyperbolic helix appears as a kind of spiral, turning infinitely many times inside the sphere $x^2+y^2+z^2=1$. Interesting drawings of this curve are given. The author goes on to discuss the common perpendicular to two skew lines in hyperbolic space and in elliptic space. He then gives a construction for the product of two displacements, using a certain skew hexagon which he boldly names "Schilling's figure." Finally, there is a "part 2" dealing with the geodesics on the cylinder (Abstandsfläche). *H. S. M. Coxeter* (Toronto, Ont.).

**Cosnitza, César. Coordonnées barycentriques.** Ann. Roumaines Math. 4, viii+176 pp. (1941).

**Coşniţă, Cezar. Sur le théorème de Carnot. Certains cas limites; extensions dans l'éspace.** Bull. Sci. Tech. Polytech. Timişoara 13, 156–165 (1948).

In the plane Carnot's formula on the points of intersection of a conic with the sides of a triangle is modified by the author to fit the case when the conic passes through one or two vertices of the triangle. The fundamental theorem of the paper is the following. Necessary and sufficient conditions that twelve points situated by twos on the six edges of a tetrahedron shall lie on the same quadric surface are that the points lying in each of the four faces shall lie on a conic. The same proposition, with somewhat weaker conditions, appeared elsewhere at about the same time [N. A. Court, Duke Math. J. 15, 49–54 (1948); these Rev. 9, 525]. The fundamental theorem is applied to a number of special cases, mostly known. The author also points out the duals of the basic proposition. *N. A. Court* (Norman, Okla.).

**Sydler, J.-P. Quelques considérations sur les sphères.** Elemente der Math. 4, 1–3 (1949).

An ordered row of spheres in which each sphere is orthogonal to the following is said to form an "orthogonal chain." If such a chain varies so that the centers of the spheres remain fixed and the orthogonality is preserved, the radical elements of any two, three, or four odd "links" of the chain remain fixed. From these considerations the author derives several propositions of which the following may be considered typical. Given four pairs of corresponding points, there exists one and only one sphere $(M)$ such that the two spheres, having for centers the points of any of the given pairs and orthogonal to the sphere $(M)$, are orthogonal to each other. *N. A. Court* (Norman, Okla.).

**Cavallaro, Vincenzo G. Sull'appartenenza di punti notevoli del triangolo alla circonferenza del suo incerchio.** Boll. Mat. (5) 2, 33–35 (1948).

**Trost, E. Zur Theorie der isoptischen Kurven.** Nieuw Arch. Wiskunde (2) 23, 4–7 (1949).

An isoptic curve $I_\alpha$ of the curve $C$ is defined as the curve described by the vertex of an angle $\alpha$ the sides of which touch $C$. All isoptic curves of a circle and a logarithmic spiral are curves similar to the original ones. On the other hand, the author remarks that for an arbitrary given curve $C$ either there are no values of $\alpha$, or there are generally only discrete values $\alpha_1, \alpha_2, \cdots$, for which $I_{\alpha_1}, I_{\alpha_2}, \cdots$ are similar to the original curve $C$. Among others the evolvent of a circle is proved to be a curve of the latter kind. *L. Fejes Tóth* (Budapest).

**Casulleras, Juan. Note on a problem of construction of quadrics.** Revista Mat. Hisp.-Amer. (4) 8, 192–194 (1948). (Spanish)

It is noted that the polar planes of a point with respect to a pencil of quadrics meet in a line, and that the aggregate of such lines is a tetrahedral complex. [The dual of this is discussed, for example, on pp. 97–102 of vol. 3 of Baker's "Principles of Geometry," Cambridge University Press, 1923.] *D. B. Scott* (London).

**Blaschke, Wilhelm. Isotrope Vierflache.** Arch. Math. 1, 182–189 (1948).

Let $x_0=1$, $x_1$, $x_2$, $x_3$ be coordinates of a Euclidean three-space. The plane $\sum_{k=0}^{3} u_{jk}x_k=0$ is called isotropic if $u_{j1}^2+u_{j2}^2+u_{j3}^2=0$ provided not all $u_{j\alpha}=0$ $(\alpha=1, 2, 3)$. A set of four isotropic planes is said to form an isotropic tetrahedron provided (1) $\sum_{k=1}^{3} u_{rk}u_{sk}\neq 0$ for $r, s=1, 2, 3$; $r\neq s$, and (2) the $4\times 4$ determinant $|u_{ik}|\neq 0$. The author chooses a convenient normalization of the variables $u_{ij}$ and uses this to derive formulae for: (1) the lengths of the edges of the tetrahedron, (2) the two axes such that reflections about these leave the tetrahedron invariant; (3) the volume of the tetrahedron, and (4) the radius of the circumsphere (there is no insphere). Relationships are developed between this geometry and the geometry of Laguerre and the dual vectors of Study. *C. B. Allendoerfer* (Princeton, N. J.).

**Pimiä, Lauri. Die konformen Involutionen des komplexen Raumes.** Soc. Sci. Fenn. Comment. Phys.-Math. 14, no. 5, 49 pp. (1948).

The points of three-dimensional complex space may be mapped on the system of circles (including straight lines and points) in the real three-dimensional space. This correspondence is used to study systems of circles as well as involutions of the complex three-dimensional space. *E. Lukacs* (China Lake, Calif.).

**Rössler, Fred. Geometrische Betrachtungen über eine Verallgemeinerung der Reliefperspektive.** Z. Angew. Math. Mech. 28, 311–316 (1948). (German. Russian summary)

The author investigates a nonlinear relief perspective. He modifies the customary procedure to obtain an image by using two coaxial right circular cylinders instead of the picture plane and the vanishing plane. The center of the projection is located on the axis of these cylinders. The paper gives a brief discussion of the properties of this mapping. *E. Lukacs* (China Lake, Calif.).

**Kustaanheimo, P. Symmetrical form of the spherical trigonometry.** Soc. Sci. Fenn. Comment. Phys.-Math. 13, no. 13, 18 pp. (1948). (English. Esperanto summary)

## *Convex Domains, Extremal Problems*

**Bernheim, B., and Motzkin, Th. A criterion for divisibility of $n$-gons into $k$-gons.** Comment. Math. Helv. 22, 93–102 (1949).

The problem as to when a convex $n$-gon in the plane can be partitioned into convex $k$-gons is given a complete solution. For $k\leqq 5$ such a partition always exists. For $k>5$, write $n-3$ in the form $(q+1)(k-2)-r$, where $0<r\leqq k-2$. Then a partition exists if and only if $r\leqq 3q$. For a fixed $n$

the number of values of $k$ for which partitions exist is finite and asymptotically equal to $(12n)^{\frac{1}{2}}$. *S. Eilenberg.*

**Süss, Wilhelm. Eine Kennzeichnung der Kugel.** Arch. Math. 1, 190–191 (1948).

If two congruent convex bodies with interior points in $E^n$ have the same center of gravity but do not coincide, then the intersection of their boundaries does not lie in a hyperplane. The sphere is the only convex body in $E^n$ which coincides with every congruent body whenever the boundaries have $n+1$ points in common which do not lie in a hyperplane. *H. Busemann* (Los Angeles, Calif.).

**Süss, Wilhelm. Über eine Affininvariante von Eibereichen.** Arch. Math. 1, 127–128 (1948).

The invariant in question has been considered by B. H. Neumann [J. London Math. Soc. 14, 262–272 (1939); these Rev. 1, 158]. The author gives a short proof of the inequalities found by Neumann and the corresponding inequalities for convex bodies of arbitrary dimension.
*W. Fenchel* (Copenhagen).

**Busemann, Herbert. A theorem on convex bodies of the Brunn-Minkowski type.** Proc. Nat. Acad. Sci. U. S. A. 35, 27–31 (1949).

The author proves the following counterpart to the Brunn-Minkowski theorem. In $n$-dimensional Euclidean space let $K$ be a convex body, $L$ an $(n-2)$-dimensional subspace which intersects $K$, and $P$ a plane normal to $L$ at a point $O$. If the $(n-1)$-dimensional volume of the intersection of any half-hyperplane $H$ bounded by $L$ is laid off from $O$ on the ray $H\cap P$, then a convex curve in $P$ results. The proof is based on ideas similar to those used in the well-known proof of the Brunn-Minkowski theorem. But the estimations by means of the latter theorem and Jensen's inequality are considerably more intricate. The author announces applications in the theory of Finsler spaces of the following corollary. Let $K$ be a convex body with center $O$. If for any hyperplane $H$ through $O$ the volume of its intersection with $K$ is laid off from $O$ on the normal to $H$ in both directions, then the resulting surface is convex.
*W. Fenchel* (Copenhagen).

**Santaló, L. A. Beweis eines Satzes von Bottema über Eilinien.** Tôhoku Math. J. 48, 221–224 (1941).

Let $K_0$ and $K_1$ be plane closed convex curves with continuous radii of curvature $R_0$ and $R_1$, and $F_{00}$, $F_{11}$, $F_{01}$ their areas and mixed area; then $F_{01}^2-F_{00}F_{11}\leqq\frac{1}{4}F_{11}^2(r_M-r_m)$, where $r_M$ and $r_m$ are the maximum and the minimum of $R_0/R_1$. This was proved by Bottema [Nederl. Akad. Wetensch., Proc. 36, 442–446 (1933)]. Using integral-geometric arguments the author obtains a new proof. *W. Fenchel.*

**Popa, Ilie. Expressions nouvelles pour la longueur des courbes et pour l'aire des surfaces fermées.** Ann. Sci. Univ. Jassy. Sect. I. 30 (1944–1947), 179–182 (1948).

The formulae in question are: $L=2\int\kappa d\sigma$, where $L$, $\kappa$, $d\sigma$ denote, respectively, the length, the curvature, the area of the sector determined by the origin and an arc element of a closed curve in the plane; $S=3\int\int HdV$, where $S$, $H$, $dV$ are, respectively, the area, the mean curvature, the volume of the sector determined by the origin and a surface element of a closed surface. From these formulae some inequalities are derived, including an upper bound for the isoperimetric deficit of an oval similar to that found by Bottema [cf. the preceding review]. [It may be mentioned that the above formulae, under suitable assumptions, are special cases of the Minkowski formula (11) in Bonnesen and Fenchel, Theorie der konvexen Körper, Springer, Berlin, 1934, p. 63.]
*W. Fenchel* (Copenhagen).

**Hadwiger, H. Notiz zur fehlenden Ungleichung in der Theorie der konvexen Körper.** Elemente der Math. 3, 112–113 (1948).

A brief report on the result communicated in a former note [Hadwiger, Glur and Bieri, Experientia 4, 304–305 (1948); these Rev. 10, 141]. *W. Fenchel* (Copenhagen).

**Trost, E. Über eine Extremalaufgabe.** Nieuw Arch. Wiskunde (2) 23, 1–3 (1949).

This is an analytical proof of the well-known elementary fact that a closed convex curve touches the sides of a circumscribed $n$-gon with minimal area at the midpoints of the sides. *L. Fejes Tóth* (Budapest).

**Bieri, Hans. Ein geometrisches Minimumproblem.** Comment. Math. Helv. 22, 103–114 (1949).

The problem considered in this paper is as follows: to find the point in space having the minimal sum of distances from $p$ given points, $q$ lines and $r$ planes. The case $q=r=0$ is a well-known classical problem treated also by several recent authors [cf., e.g., Weiszfeld, Tôhoku Math. J. 43, 355–386 (1937)]. A complete solution of the problem is not obtained. *L. Fejes Tóth* (Budapest).

**Zalgaller, V. A. A problem on the maximum ratio of the distance on a surface to the distance in space.** Uspehi Matem. Nauk (N.S.) 3, no. 3(25), 202–207 (1948). (Russian)

The author imposes on a convex surface $F$ two conditions: (1) that it be contained in a sphere $S$ and (2) that the volume bounded by it be not greater than $V$, and considers the ratio of the distance between two points of $F$ measured on $F$ to the distance between the same points measured in space. He proves that this ratio reaches its maximum when $F$ is a symmetric plate cut out of $S$ by two parallel planes (and when the points are the centers of the flat faces of $F$). If condition (2) is replaced by the condition that $F$ should contain a sphere $s$ concentric with $S$ the result remains essentially the same; if the restriction that $s$ be concentric with $S$ is dropped the surface $F$ for which the ratio is maximum is still cut out from $S$ by two planes but they are no longer parallel, and under certain conditions involving a root of a complicated transcendental equation $F$ may be a wedge. The author also gives without proof an analogous result for the case when condition (2) is replaced by the condition that $F$ should contain in every direction a segment of length $a$. *G. Y. Rainich* (Ann Arbor, Mich.).

**Pogorelov, A. V. Rigidity of convex surfaces.** Doklady Akad. Nauk SSSR (N.S.) 62, 27–29 (1948). (Russian)

Let convex surface denote a connected open subset $S$ of the boundary of a convex body in $E^3$. The greatest lower bound of the lengths of all curves in $S$ connecting two points $a$, $b$ of $S$ in the distance of $a$ and $b$. Two convex surfaces $S$, $S'$ are isometric if a distance-preserving mapping of $S$ on $S'$ exists. The surface $S$ is called rigid if every convex surface isometric to $S$ is congruent to $S$, that is, can be carried into $S$ by a motion of $E^3$, reflection admitted. The surface $S$ has bounded curvature if the ratio of the spherical excess of a geodesic triangle to its area is uniformly bounded.

Closed convex surfaces of bounded curvature are rigid. If a convex surface $S$ is bounded (in $E^3$) and has a boundary which consists of a finite number of curves, each with total geodesic curvature $2\pi$ (in the generalized sense of A. D. Aleksandrov), then $S$ is rigid. A convex cap is a convex surface with a plane boundary and such that different points of the cap have different projections on the plane. Convex caps are rigid among caps. Let $F$ be the total boundary of a nonbounded convex body in $E^3$. Denote by $K(R)$ the geodesic circle with radius $R$ about a fixed point $O$ of $F$, by $\omega(R)$ the total geodesic curvature of $K(R)$ and by $l(R)$ the length of the shortest closed curve on $F$ that contains $K(R)$. If $F$ has bounded curvature and $[2\pi-\omega(R)]/l(R)\to 0$ for $R\to\infty$, then it is rigid. Also, some results of S. Olovianishnikoff [Rec. Math. [Mat. Sbornik] N.S. **18(60)**, 429–440 (1946); these Rev. **8**, 169] are generalized. There is only an indication of the proof of the first theorem.

*H. Busemann* (Los Angeles, Calif.).

**Pogorelov, A. V. A general theorem on infinite convex polyhedra.** Doklady Akad. Nauk SSSR (N.S.) **62**, 167–169 (1948). (Russian)

A system $v_1, \cdots, v_k$ of vectors in $E^n$ pointing into the half-space $x_n>0$ is called convex if no $v_s$ is a linear combination, with nonnegative coefficients, of the remaining $v_j$. Denote by $V$ the (conical) set of points representable in the form $\sum\lambda_i v_i$, $\lambda_i\geqq 0$. Let $\omega$ be a nonnegative function defined on all $(n-1)$-dimensional (solid) convex polyhedra $Q$ and satisfying the three conditions: (1) if $Q'$ can be obtained from $Q$ by a translation of $E^n$ then $\omega(Q')=\omega(Q)$; (2) if $Q'$ is properly contained in $Q$, then $\omega(Q')<\omega(Q)$; (3) if $s$ is the $(n-1)$-dimensional volume of $Q$, then continuous functions $c_1(s)$ and $c_2(s)$ with $c_1(s)<\omega(Q)<c_2(s)$ exist such that $c_2(s)\to 0$ for $s\to 0$ and $c_1(s)\to\infty$ for $s\to\infty$. The following generalization of a theorem of Minkowski is given. Let $v_1, \cdots, v_k$, $k\geqq n$, be a convex system of vectors, $l_1, \cdots, l_m$ a system of rays in the corresponding set $V$, moreover, $h_1, \cdots, h_k$ any real numbers, and $p_1, \cdots, p_m$ any positive numbers. There exists a unique (unbounded) convex polyhedron $\Pi$, whose infinite faces have normals parallel to $v_1, \cdots, v_k$, whose finite faces have normals parallel to $l_1, \cdots, l_m$ and such that $H(v_i)=h_i$, $i=1, \cdots, k$, where $H$ is the supporting function of $\Pi$ and $\omega$ has the value $p_j$ on the face with normal $l_j$.

*H. Busemann* (Los Angeles, Calif.).

## *Algebraic Geometry*

***Hodge, W. V. D., and Pedoe, D. Methods of Algebraic Geometry.** Vol. I. Cambridge, at the University Press; New York, The Macmillan Company, 1947. viii+440 pp. $6.50.

The material covered in this book contains the essential prerequisites for a first course in algebraic geometry, namely certain well-defined topics in modern algebra and the general theory of projective spaces. Accordingly this volume is divided into two parts, entitled respectively "Algebraic preliminaries" and "Projective space." The book is presented by the authors as the first part of a treatise on algebraic geometry and is intended to clear the way for a forthcoming second volume which will be devoted "to algebraic varieties and to the study of certain loci which arise in many geometric problems."

The first chapter on "Rings and fields" deals with the fundamental concepts of pure algebra. "Linear algebra, matrices, determinants" are treated in the second chapter. In view of later applications to general projective spaces, a good deal of this material is developed for noncommutative as well as for commutative fields. Chapter III on "Algebraic dependence" deals with the fundamentals of field theory (transcendental and algebraic field extensions). This topic, which is often mentioned only briefly in a first course in algebra, is, however, paramount for algebro-geometric purposes. In the next chapter the authors approach the topic of "Algebraic equations" from the side of elimination theory. Resultants and resultant systems are studied in great detail, and the results are applied toward the derivation of Hilbert's Nullstellensatz. This chapter concludes the algebraic part of the book.

Chapter V begins with the introduction of the projective number space and proceeds from that to the algebraic definition of the general projective space, the principle of duality, and the theorem of Desargues. On the basis of the graphical constructions of the sum and product of points it is then proved that the theorem of Pappus is equivalent to the commutative law of multiplication. Further implications of Pappus's theorem are discussed in the next chapter which is devoted to a synthetic definition of projective spaces. The graphical constructions mentioned above now serve as a basis for developing the algebra of points and for introducing a system of coordinates. The last three chapters deal respectively with "Grassmann coordinates," "Collineations" and "Correlations." The chapter on Grassmann coordinates includes a proof that the well-known quadratic relations between these coordinates form a basis of the homogeneous prime ideal of the corresponding Grassmann variety.

*O. Zariski* (Cambridge, Mass.).

**Zariski, Oscar. A simple analytical proof of a fundamental property of birational transformations.** Proc. Nat. Acad. Sci. U. S. A. **35**, 62–66 (1949).

The property referred to is contained in the "main theorem" of the author's paper on birational transformations [Trans. Amer. Math. Soc. **53**, 490–542 (1943); these Rev. **5**, 11], which states that if a birational correspondence $T$ exists between two irreducible varieties $V$ and $V'$ in which there are no fundamental elements on $V'$, and if $W$ is an irreducible fundamental variety of dimension $r$ on $V$ at which $V$ is locally normal, then any isolated component of the transform $T[W]$ of $W$ is of dimension greater than $r$. The new proof is based on certain properties of local rings which are established. Using these, it is shown that any valuation $v$ of the common function field of $V$ and $V'$, whose centre on $V$ is $W$, can be extended to give a valuation of the quotient field of the completion $O^*$ of the local ring of $W$. If $W'$ is an isolated component of $T[W]$ of dimension $r$, it is shown that the local ring of $W'$ is integrally dependent on $O^*$, and that this implies that the local ring of $W'$ is contained in the valuation ring of $v$. Hence the centre of $v$ on $V'$ contains $W'$. Since $W'$ is an isolated component of $T[W]$, the centre of $v$ on $V'$ is necessarily $W'$; that is, $W'=T[W]$. This implies that $W$ is not fundamental and the theorem follows.

*W. V. D. Hodge* (Cambridge, England).

**Chow, Wei-Liang. On the geometry of algebraic homogeneous spaces.** Ann. of Math. (2) **50**, 32–67 (1949).

Let $S_n$ be the projective space of dimension $n$ over an arbitrary basic field $K$. If $m\leqq n$, let $G_{n,m}$ be the space of all

$m$-dimensional subspaces of $S_n$. Let $\Delta$ be an involutory correlation in $S_n$. Then $\Delta$ associates to every $s$-dimensional subspace $H$ of $S_n$ a conjugate space $\bar{H}$ of dimension $n-s-1$; $H$ is said to be invariant if either $H\subset\bar{H}$ or $\bar{H}\subset H$. Set $r=(n-1)/2$ if $n$ is odd, $r=(n-2)/2$ if $n$ is even, and denote by $J_r$ the space of invariant subspaces of dimension $r$ of $\Delta$.

If $M$ is either one of the spaces $G_{n,m}$ or $J_r$, there is a group $\Gamma$ which operates naturally on $M$: if $M=G_{n,m}$, $\Gamma$ is the group of collineations of $S_n$ (or of collineations and correlations if $n=2m+1$); if $M=J_r$, $\Gamma$ is the group of those collineations of $S_n$ which preserve $\Delta$. In any case, $\Gamma$ is to be called the basic group of $M$. The object of the paper is to give various characterizations of those one-to-one mappings of $M$ onto itself which are produced by operations of the basic group. In the case where $M=J_r$, it will be assumed that $M$ contains at least two disjoint $r$-dimensional subspaces of $S_n$, and that not all elements of $M$ lie on some hyperplane of $S_n$.

In the first part of the paper, projective characterizations of the operations belonging to $\Gamma$ are given. Two subspaces of dimension $m$ of $S_n$ are said to be adjacent to each other if their intersection is of dimension $m-1$. If $n-1>m>0$, then any one-to-one transformation of $G_{n,m}$ onto itself which transforms any pair of adjacent spaces into a pair of adjacent spaces belongs to the basic group. If $M=J_r$, then any one-to-one transformation of $M$ onto itself which transforms any pair of adjacent elements of $M$ into a pair of adjacent elements belongs to the basic group if $r>1$. The restriction $r>1$ may be omitted if $\Delta$ defines a null system. Let us say that two invariant points of $S_n$ are co-adjacent if the line which joins them is invariant. Then any one-to-one transformation onto itself of the set of invariant points which transforms any pair of coadjacent points into a pair of coadjacent points is induced by a collineation if $n>4$, and even in the case $n=4$ in the case of a polar system. This means that, if $Q$ is a hyperquadric in canonical form in a projective space of dimension $n>3$, then any one-to-one mapping of $Q$ onto itself which transforms any straight line on $Q$ into a straight line on $Q$ is induced by a collineation of $S_n$.

In the case of a polar system, $J_r$ can be decomposed into two irreducible spaces. Let $J_r'$ be one of them. Call now "adjacent" two elements of $J_r'$ whose intersection is of dimension $r-2$. Then any one-to-one transformation of $J_r'$ onto itself which carries any pair of "adjacent" elements into a pair of "adjacent" elements belongs to the basic group if $r>3$. But this is not true any more if $r=3$.

The second part of the paper provides algebraic characterizations of the operations of $\Gamma$ (or rather of those which do not involve automorphisms of the basic field). It is assumed that $M$ is either $G_{n,m}$ or the space of invariant elements of a null system, or one of the two irreducible spaces of invariant elements of a polar system, or a nondegenerate hyperquadric in $S_n$. The space $M$ is then put in a one-to-one correspondence with the set of points of an algebraic variety $M'$ in a projective space. Then it is proved that any everywhere regular birational transformation of $M'$ onto itself is an element of the basic group, except in the case where $M=J_3'$. The proof is based on the determination of the complete linear systems on $M'$ and on the projective characterizations obtained in the first part.

In the third part of the paper, the basic field $K$ is assumed to be that of complex numbers. The variety $M'$ of the second part may then be considered as a complex manifold, and it is proved by a simple topological argument that any regular analytic transformation of $M'$ onto itself is a transformation of the basic group (with, naturally, the same exception as in the second part). *C. Chevalley.*

**Segre, Beniamino. Sui teoremi di Bézout, Jacobi e Reiss.** Ann. Mat. Pura Appl. (4) **26**, 1–26 (1947).

The author takes up the systematic study of necessary and sufficient conditions that a set of $n\nu$ points in the plane be the intersection of two algebraic curves of orders $n$ and $\nu$. If $G_{n\nu}$ is such a set of $n\nu$ distinct points and $n\geqq 3$ or $\nu\geqq 3$, then the known result that these points impose at most $n\nu-1$ conditions on the $C_{n+\nu-3}$ constrained to contain them is easily derived. Inversely, a set $G_{n\nu}$ of $n\nu$ distinct points thus associated with the $C_{n+\nu-3}$ and lying on an irreducible $C_n$ is the set of intersections of this $C_n$ and a $\Gamma_\nu$. The way (limit process) is indicated for extending these results to the general case in which some of the $n\nu$ points are infinitely near to others, but this is not insisted upon and some special cases to be applied immediately are separately derived without recourse to the notion of limit: in particular, the case of $(l+1)n\nu$ points which constitute the complete intersection on a $C_n$ of a $\Gamma_\nu$ counted $l+1$ times is considered, where $C_n$ and $\Gamma_\nu$ intersect in $n\nu$ distinct points. From this point on, the Abelian integral

$$\mathfrak{J}=\int_C \frac{\alpha(x,y)dx}{f_y(x,y)\varphi^{l+1}(x,y)}$$

is made the center of the argument, where $f(x,y)=0$ is $C=C_n$, $\varphi(x,y)=0$ is $\Gamma$, $C$ and $\Gamma$ intersect in $n\nu$ distinct points at finite distance, and $\alpha(x,y)$ is a polynomial of degree not exceeding $n+\nu-2$. Applying the fact that the sum of the polar periods of this integral is zero, one finds for $l=0$ that $\sum_{j=1}^{n\nu}\alpha(x_j,y_j)/J(x_j,y_j)=\sum_{i=1}^{n}\alpha^*(1,t_i)/f_y^*(1,t_i)\varphi^*(1,t_i)$, where $(x_j,y_j)$, $j=1,\cdots,n\nu$, are the intersections of $C$ and $\Gamma$, $J=\partial(\varphi,f)/\partial(x,y)$, $f^*(x,y)$ and $\varphi^*(x,y)$ are the terms of degree $n$ and $\nu$ of $f$ and $\varphi$, the $t_i$ are the roots of $f^*(1,t)=0$, and $\alpha^*(x,y)$ are the terms of degree $n+\nu-2$ of $\alpha(x,y)$. For $\alpha$ of degree not exceeding $n+\nu-3$, i.e., $\alpha^*=0$, one obtains the equation of Jacobi: $\sum_{j=1}^{n\nu}\alpha(x_j,y_j)/J(x_j,y_j)=0$. By appropriate choice of $\alpha$, one may derive the theorem of Humbert on the sum of the cotangents of the angles under which $C$ and $\Gamma$ meet, namely, that this sum depends only on the points where $C$ and $\Gamma$ intersect the line at infinity.

For $l=1$, one sees from what has been said that there is a relation (in fact, there is only one) binding the second (and lesser) derivatives of $y$ and $\eta$ at the points $P_j:(x_j,y_j)$, where $y=y(x_j)$, $y=\eta(x_j)$ represent $C$ and $\Gamma$ near $P_j$, the $y$-axis supposed not parallel to the tangents of $C$ and $\Gamma$ at $P_j$, $j=1,\cdots,n\nu$. For $\Gamma$ a straight line, the relation $\sum_{i=1}^{n}\kappa_i\csc^3\tau_i=0$, where $\kappa_i$ is the curvature of $C$ at $P_i$ and $\tau_i$ is the angle which $C$ makes with $\Gamma$, was first obtained by M. Reiss in 1837, and was subsequently forgotten and rediscovered several times. Using the integral $\mathfrak{J}$ with $l=1$, one finds more generally that

$$\sum_{j=1}^{n\nu}(y_j''(x_j)-\eta_j''(x_j))/(y_j'(x_j)-\eta_j'(x_j))^3=0.$$

In the case of $\nu=1$, the theorem of Reiss is also generalized to include nondistinct (i.e., infinitely near) intersections of $C$ with the straight line $\Gamma$. The various results above mentioned (of Jacobi, Humbert, Reiss) are also generalized to hyperspace. In another note [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 167–172, 172–179 (1947); these Rev. 9, 527] the author frees the argument from the theory of Abelian integrals. *A. Seidenberg.*

**Masotti Biggiogero, Giuseppina. Precisazione di singolarità della hessiana.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **10(79)**, 89–96 (1946).

L'auteur apporte une précision complémentaire à une étude antérieure qu'elle a faite sur le même sujet [même Rend. Cl. Sci. Mat. Nat. (3) **5(74)**, 317–324 (1941); **7(76)**, 271–280 (1943); ces Rev. **8**, 343]. Si une courbe plane algébrique $F$ admet un point $O$ tacnodal d'espèce $r-1$, la Hessienne présente en $O$ trois branches linéaires ayant entre elles et avec $F$ un contact $r$-ponctuel. Si une courbe plane algébrique $F$ admet un point $O$ comme rebroussement d'espèce $r$, la Hessienne présente en $O$ deux branches: l'une, linéaire, passe par les $r$ points doubles et le premier point simple constitutif du rebroussement; l'autre est une branche du second ordre qui a en commun avec $F$ les $r$ points doubles, le premier point simple consécutif et son satellite.

*L. Gauthier* (Nancy).

**Tibiletti, Cesarina. Sulle curve triple prive di punti di diramazione.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **10(79)**, 227–244 (1946).

Construire une courbe multiple $f(x, y)=0$ sans diramations, c'est construire une fonction $z(x, y)$ sans diramation sur la Riemannienne $R_f$. Dans le cas d'une courbe triple, on associe à $z$ la fonction $w$ donnée par la résolvante de Galois, d'ordre 6 à laquelle se lie un groupe diédrique isomorphe aux mouvements de l'hexagone régulier. Pour passer de la Riemannienne $R_f$ à celle $R_z$ de $z$, on peut extraire un radical carré $v$ sur $R_f$ d'où $R_v$ puis sur $R_v$ extraire une racine cubique $w$ d'où $R_w$; en associant 2 à 2 les valeurs de $w$ on construit $z$. L'utilisation des Riemanniennes intermédiaires $R_v$ et $R_w$ montre que les diramations de $z$ et $w$ correspondent à celles de $v$ sur $R_f$ et de $w$ sur $R_v$. L'auteur montre alors en se servant de la représentation hexagonale du groupe que les $z$ sans diramations correspondent complètement aux $w$ à groupe diédrique irréductible sans diramations en sorte qu'à des $z$ distinctes correspondent des $w$ distinctes et inversement; les $w$ s'obtiennent par extraction du radical $v$ sur $R_f$ et du radical $w$ sur $R_v$ de toutes les façons possibles birationnellement distinctes. Ceci ramène à la construction de Chisini [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **24**$_1$, 153–158 (1915)] pour les groupes cycliques donc à des choix de substitution le long des cycles de la Riemannienne.

Par de simples combinaisons sur les cycles diramants ou non, l'auteur construit une forme canonique de $R_f$ avec rétrosection telles que $v$ irréductible sans diramation sur $f$ ne dirame que le long d'une rétrosection; à partir de cette $R_f$, elle construit une $R_v$ de genre $2p-1$ formée d'un tore à $2p-2$ manches, les rétrosections du tore étant l'une autoconjuguée dans l'involution $v$, l'autre conjuguée d'un cycle homologue, les manches et leurs rétrosections étant 2 à 2 conjugués. Sur $R_f$ on choisit arbitrairement les substitutions chacune donnant une $v$ distincte; sur $R_v$ le groupe diédrique impose le choix de substitutions réciproques sur des cycles conjugués. Il en résulte que si $p=1$ on ne peut diramer sur les cycles associés, d'où impossibilité de construire le groupe diédrique irréductible. En conséquence on ne peut construire une courbe elliptique triple générale sans diramations.

*B. d'Orgeval* (Grenoble).

**Conforto, F. Sopra un caso particolare della superficie $F_4^{(3)}$ di M. Noether.** Experientia **4**, 382–383 (1948).

Noether a montré jadis [M. Noether, Math. Ann. **33**, 546–571 (1889)] qu'une surface $F_4$ pouvait être rationnalisée par un point triple, un tacnode, ou un point double uniplanaire $O$ par lequel passe une droite $r$ de la surface telle que les plans contenant $r$ coupent selon un faisceau de cubiques dont $r$ est tangente d'inflexion; le plan tangent en $O$ coupe selon $r$ (deux fois) et une conique $C_2$. La représentation plane de la surface se fait par le système des $C_9(8A^3, A_9^2, A_{10})$ dont les points-base appartiennent à une cubique fondamentale $C_3$, image du point singulier; $A_{10}$ représente $r$ et $A_9$, $C_2$. L'auteur étudie le cas où les 9 premiers points-base sont bases d'un faisceau d'Halphen de $C_9$. Alors $A_{10}$ doit être infiniment voisin de $A_9$, d'où résulte que la conique se décompose en la droite $r$ et une autre droite $t$ ne passant pas en $O$; les plans par $t$ coupent selon un second faisceau de cubiques représentées par le faisceau de Halphen. Dans ce cas, la surface possède deux faisceaux de cubiques.

*B. d'Orgeval* (Grenoble).

**Seifert, L. L'hypersurface cubique ayant un point bispatial dans l'espace à quatre dimensions.** Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **46** (1945), 73–76 (1946).

Summary of the Czech original which appeared in Rozpravy II. Třídy České Akad. **55**, no. 7 (1945); these Rev. **9**, 375.

**Seifert, L. L'hypersurface cubique à six points doubles dans l'espace à quatre dimensions et quelques figures géométriques liées à deux courbes cubiques situées sur une surface du second ordre.** Publ. Fac. Sci. Univ. Masaryk no. **287**, 15 pp. (1947). (Czech. French summary)

## *Differential Geometry*

**Kasner, Edward, and De Cicco, John. Physical systems of curves in space.** Proc. Nat. Acad. Sci. U. S. A. **35**, 106–108 (1949).

This is, in effect, a continuation of a previous note by the same authors [same Proc. **34**, 68–72 (1948); these Rev. **9**, 375]. Some additional theorems relating to physical systems of curves in three-dimensional space are stated.

*L. A. MacColl* (New York, N. Y.).

**Myller, A. La discrimination des podaires des courbes planes selon leurs propriétés géométriques.** Ann. Sci. Univ. Jassy. Sect. I. **30** (1944–1947), 171–178 (1948).

Properties of the pedal curve $P$ of a curve $C$ relative to a pole are studied with emphasis upon the distinctions among the various pedal curves. Three types of mean radii for $P$ are discussed: in $R_1$ the average is taken over polar angles of $P$; in $R_2$ over arc length; in $R_3$ over the areas of sectors described by the radius of $P$. The following results are proved. (1) All pedals have the same $R_1$. (2) If $R_3$ is constant, the pole must lie on a certain family of circles (Steiner's theorem). (3) The area of $P$=area of $C$+area of the counterpedal (Catalan's theorem). (4) If $R_3$ is constant the pole lies on a certain pencil of conics. (5) If the geometric mean of $R_2$ and $R_3$ is constant the pole lies on a family of conics. The differential equation of the set of pedal curves of $C$ is obtained, and certain types of pedals are discussed in terms of the solutions of this equation.

*C. B. Allendoerfer* (Haverford, Pa.).

**Herzog, F., and Wells, C. P. A problem concerning orthogonal trajectories.** Quart. Appl. Math. **7**, 121–126 (1949).

Consider a mapping $u=u(x, y)$, $v=v(x, y)$ of a region of the $(x, y)$-plane upon a region of the $(u, v)$-plane. The real

functions $u(x, y)$ and $v(x, y)$ are assumed to have continuous second derivatives with Jacobian not zero. Let $A$ and $B$ denote the families of curves in the $(u, v)$-plane which are the images of the parallel pencils of lines $y$=constant and $x$=constant, respectively. For two orthogonal families $A$ and $B$, the family $A$ is said to be of proportional arc length if the following condition is fulfilled. Let $y_1$, $y_2$, $y_3$ be any three curves of $A$, and let $s_1$ be the arc length of any curve of $B$ between $y_1$ and $y_2$ and $s_2$ the arc length of the same curve between $y_2$ and $y_3$. Then the ratio $s_1/s_2$ is to be constant for all curves of $A$, that is, dependent only on the choice of $y_1$, $y_2$, $y_3$. The following three theorems are established. (1) If $A$ and $B$ are orthogonal families, then $A$ is of proportional arc length if and only if $u_x^2+v_x^2$ is separable, that is, is the product of a function of $x$ by a function of $y$. Similarly $B$ is of proportional arc length if and only if $u_y^2+v_y^2$ is separable. (2) If two families $A$ and $B$ of orthogonal trajectories are both of proportional arc length, then they must both be isothermal families. Conversely, if one of two families of an isothermal net is of proportional arc length, so is the other. (3) The totality of different types of isothermal nets of proportional arc length are those obtained from the following analytic functions: (a) $w=z$; (b) $w=e^z$; (c) $w=\exp(ze^{i\gamma})$, where $0<\gamma<\pi/2$; (d) $w=\int_0^z \exp(-t^2)dt$. Physical interpretations of these results are given.

*J. De Cicco* (Chicago, Ill.).

**Hsiung, Chuan-Chih. A graphical construction of the sphere osculating a space curve.** Tôhoku Math. J. **48**, 272–276 (1941).

A purely geometrical construction is derived for the osculating sphere of a skew curve $C$ at one of its points. The curve is assumed to be given in terms of its projections $C_1$, $C_2$ on two perpendicular planes. The construction is given in terms of the centers of curvature of $C_1$, $C_2$ of their evolutes, and certain lines and points determined by them. The proof of the construction is based on elementary considerations, the equations of $C$ being in the Monge form.

*V. G. Grove* (East Lansing, Mich.).

**Rollero, Aldo. Un'osservazione sugli inviluppi.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **10(79)**, 69–70 (1946).

Let $C$ be a plane curve and $P$ a stationary or multiple point of $C$. It is pointed out that the tangent or tangents to $C$ at $P$ may be considered as forming part of the envelope of the tangents to $C$. *V. G. Grove* (East Lansing, Mich.).

**Rollero, Aldo. Sul contatto del terzo ordine di due superficie in un loro punto.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **10(79)**, 112–116 (1946).

A study is made of two surfaces having third order contact at a point. The main result of the paper may be stated as follows. A necessary and sufficient condition that two surfaces $S$ and $S'$, with the same asymptotic tangents, either distinct or coincident, at a common point $O$ have third order contact at $O$ is that they have coincident tangent planes at any two distinct points belonging to the neighborhoods of the second order of these surfaces at $O$.

*V. G. Grove* (East Lansing, Mich.).

**Sbrana, F., e Rollero, A. Su una proprietà elementare dei raggi di curvatura delle curve e delle superficie.** Atti Accad. Ligure **4** (1947), 16–20 (1948).

Attention is called to an elementary property of the principal radii of normal curvature $R_1$, $R_2$ of a surface $S$, with $R_2>R_1>0$. Let $P$ be a point on the normal to $S$ at $O$, at a numerical distance of $r$ from $O$, and let $A$ be a point on $S$ near $O$, at a numerical distance of $d$ from $A$. Then $d$ has a minimum value if $r\leqq R_1$, has a maximum value if $r\geqq R_2$, and has neither a maximum nor a minimum if $R_1<r<R_2$.

*V. G. Grove* (East Lansing, Mich.).

**Rollero, Aldo. Sui punti delle superficie rigate.** Atti Accad. Ligure **4** (1947), 43–49 (1948).

Canonical expansions are determined for a nondevelopable ruled surface at an ordinary point, and at a point of one of its flecnode curves. These expansions have the respective forms $z=xy-y^4+[5]$, $z=xy+y^2(x+y)+[5]$, where $[5]$ indicates terms of the fifth degree and higher. A geometrical characterization of the latter expansion is given in terms of quadrics and cubics having third order contact with the surface. The method is that of Wilczynski and Lane in their studies of ruled surfaces. *V. G. Grove.*

**Rollero, Aldo. Sui punti flecnodali e biflecnodali delle superficie.** Atti Accad. Ligure **4** (1947), 57–64 (1948).

Canonical expansions are found for a nonruled surface at flecnodal and at biflecnodal points. These expansions have the respective forms $z=xy-y^3+I(x-4y)x^3+[5]$, $z=xy+xy(x^2+y^2)+Px^4+Sy^4+[5]$, $I, P, S$ being invariants. A partial geometrical interpretation is given for the latter expansion, similar to that in the paper reviewed above. The method is that of Wilczynski for nonruled surfaces.

*V. G. Grove* (East Lansing, Mich.).

**Saban, G. Alcune limitazioni integrali nella teoria metrica delle rigate.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 24–30 (1948).

B. Segre [same Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 422–426 (1947); these Rev. **10**, 208] has given an estimate of the total torsion of a closed skew curve. In exactly the same way and by means of Segre's lemma the author obtains analogous results for the real parts of the total dual curvature and the total dual torsion of a closed ruled surface. [For these notions cf. Blaschke, Vorlesungen über Differentialgeometrie, v. 1, 3d ed., Springer, Berlin, 1924, § 121].

*W. Fenchel* (Copenhagen).

**Saban, G. Alcune limitazioni integrali nella teoria metrica delle congruenze rettilinee.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 134–139 (1948).

In the geometry of Study's dual vectors there is an analogue of the Gauss-Bonnet theorem which is stated without proof by W. Blaschke [Vorlesungen über Differentialgeometrie, v. 1, 3d ed., Springer, Berlin, 1930, p. 281]. The author applies this result to obtain the following theorems. (1) If a line congruence is bounded by a closed general regulus $R$, then $-L\leqq 2\int\bar{\delta}d\omega\leqq 0$, where $L$ is the length of the striction curve of $R$, $\bar{\delta}$ is the mean distribution parameter at a line of $C$ and $d\omega$ is the surface element of the spherical indicatrix of the lines of $C$. (2) If the congruence $C$ is closed, $\int\bar{\delta}d\omega=0$.

The definition of a "general" regulus appears in the author's paper [see the preceding review], the results of which are used in the present paper. Unfortunately the author does not mention his assumption of the generality of the regulus in (1) above; if this assumption is relaxed, the formula of (1) must be rewritten $L\geqq|2\int\bar{\delta}d\omega|$. In proving (2) he assumes implicitly that the indicatrix of the congruence covers the sphere exactly once, but this does not appear to be a restriction on the validity of the theorem.

*C. B. Allendoerfer* (Haverford, Pa.).

**Lalan, Victor. Sur les vecteurs indicatifs d'un réseau quelconque tracé sur une surface.** C. R. Acad. Sci. Paris **228**, 294–296 (1949).

Let $\lambda$ and $\mu$ be two vector fields of arbitrary length which are respectively tangent to the curves of a net defined on a surface. The author introduces the indicatrices $P_\alpha = \lambda_{\alpha,\beta}\mu^\beta - \mu_{\alpha,\beta}\lambda^\beta$ and $Q^\rho = \epsilon^{\alpha\rho}P_\alpha/\sqrt{g}$. These are interpreted in terms of invariants of Graustein [Bull. Amer. Math. Soc. **36**, 489–521 (1930)]. *C. B. Allendoerfer.*

**Lalan, Victor. Sur les surfaces à courbure moyenne isotherme.** Canadian J. Math. **1**, 6–28 (1949).

A surface of isothermal mean curvature (called an HI for short) is one on which the curves of constant mean curvature are isothermal lines. Among such surfaces are those admitting a one-parameter group of motions (cylinders, surfaces of revolution, helicoids) and surfaces admitting an infinite family of deformations with preservation of the principal curvatures (surfaces of O. Bonnet). This paper is a connected treatment of the author's study of such surfaces and is a sequel to his earlier summary paper [Bull. Soc. Math. France **75**, 63–88 (1947); these Rev. **9**, 375].

In chapter I the two fundamental forms of the surface are written $ds^2 = 2\omega_1\omega_2/A$; $\Phi = \omega_1^2 + 2H\omega_1\omega_2/A + \omega_2^2$, where $H$ is the mean curvature and $A$ the asphericity. Two additional invariants, $r$ and $s$, are defined by $d\omega_1 = r[\omega_1\omega_2]$; $d\omega_2 = s[\omega_2\omega_1]$. It is shown that the equations of Codazzi may be written $dH = -2A(r\omega_1 + s\omega_2)$. A so-called "primitive function" $\psi(u, v)$ of HI exists such that $\omega_1 = (\psi_u)^{\frac{1}{2}}du$, $\omega_2 = (\psi_v)^{\frac{1}{2}}dv$ and $H = f(u+v)$. This is systematically used to express properties of HI. The curves $\psi(u, v) = C$ are called "primitive curves." It is shown that on HI the primitive curves and the lines of constant mean curvature are bisected by the lines of curvature. A fifth order differential equation is obtained for surfaces HI.

Chapter II discusses surfaces HI which moreover are isothermal. The chief result is that a necessary and sufficient condition that HI be isothermal is that the primitive function can be written $\psi = g(U+V)$. In chapter III an isothermal HI is further assumed to be $W$, namely a surface for which there is a relation between $H$ and $A$. It is proved that the only surfaces having all these properties are surfaces of revolution, cylinders, certain cones, and surfaces of O. Bonnet of the third class. These methods are used in chapter IV to prove that, if a $W$ surface (of variable total and mean curvature) is applicable on a surface of revolution, it is either a helicoid, a surface of revolution, or a surface of O. Bonnet. The paper concludes [chapter V] with some remarks on surfaces of O. Bonnet. Their differential equations are obtained, and it is proved that a necessary and sufficient condition that a surface be of this type is that it be an HI whose primitive function has its two parameters proportional. The three classes of such surfaces are defined and detailed properties are developed.

*C. B. Allendoerfer* (Haverford, Pa.).

**Backes, Fernand. Sur certains couples de surfaces dont les tangentes asymptotiques se coupent inversement.** C. R. Acad. Sci. Paris **228**, 49–51 (1949).

Let $S_1$, $S_2$ be two surfaces in one-to-one correspondence, their asymptotic curves $u$ = constant, $v$ = constant corresponding. Let $P_1$, $P_2$ and $\pi_1$, $\pi_2$ be their corresponding points, and tangent planes at those points. The following theorem is proved. If the tangent lines to $u$ = constant, $v$ = constant at $P_1$ intersect respectively the tangent lines to $v$ = constant and $u$ = constant at $P_2$, then if $S_1$, $S_2$ have one of the three following properties they have the other two: (1) projectively applicable; (2) $P_1P_2$ generates a $W$ congruence; (3) $\pi_1\pi_2$ generates a $W$ congruence. The method used is that of Cartan. *V. G. Grove* (East Lansing, Mich.).

**Finikoff, S. Couple de courbes stratifiables.** Univ. Nac. Tucumán. Revista A. **6**, 289–312 (1948).

Let a plane element $E(P, \pi)$ be defined by a plane $\pi$ and a point $P$ in $\pi$. The element $E(P, \pi)$ is said to be stratifiable if there exist $\infty^1$ surfaces tangent to each plane $\pi$ of $E$ at $P$. For example, let there be two congruences $K(r)$, $K'(r')$ whose lines $r$, $r'$ are in one-to-one correspondence. Let $P$ be on $r$, and $\pi$ the plane determined by $P$ and $r'$; similarly let $P'$ be on $r'$, and $\pi'$ the plane of $P'$ and $r$. Then these congruences $K$ and $K'$ are said to be stratifiable if $E(P, \pi)$ and $E'(P', \pi')$ are stratifiable.

The paper is concerned with applying this idea to curves $C$, $C'$ in one-to-one point correspondence. The plane elements $E$, $E'$ are determined as follows. Let $M$ and $M'$ be corresponding points of $C$ and $C'$, and $\sigma$ and $\sigma'$ the osculating planes of $C$ and $C'$ at $M$, $M'$. Let there be established in some geometrically meaningful manner a correlation $F$ between $\sigma$ and $\sigma'$, so that to a point $P$ of $\sigma$ corresponds a line $p'$ of $\sigma'$, and to each point $P'$ of $\sigma'$ a line $p$ of $\sigma$. Then the line elements used are $E(P, \pi)$, $E'(P', \pi')$, $\pi$ being the plane of $P$ and $p'$ and $\pi'$ that of $P'$ and $p$. Then $C$ and $C'$ are said to be stratifiable relative to the correlation $F$ if $E$ and $E'$ are stratifiable by the above definition. The paper considers the case in which $C$, $C'$ have a common osculating linear complex $K$. The correlation $F$ is that determined by the null system established by a tangent complex $K'$ which is in involution with $K$. The surfaces $\Sigma$, $\Sigma'$ which touch $\pi$ and $\pi'$ at $P$ and $P'$ are ruled surfaces whose generators form two pencils of lines with centers at $M$, $M'$ in the planes $\sigma$, $\sigma'$, respectively. The curves $C$, $C'$ thus give rise to a system of $\infty^2$ congruences $K$, $K'$, having the $\infty^1$ surfaces $\Sigma$, $\Sigma'$ as focal surfaces. These congruences are $W$ congruences.

Each osculating plane $\sigma$ of $C$ contains homologous generators of the surfaces $\Sigma$, and all these generators pass through $M$ where $\sigma$ touches $C$. Each point of $C$ belongs to all the surfaces $\Sigma$. The plane $\sigma$ corresponds to $M$ in the null system of $K$; hence $\sigma$ is tangent to all surfaces $\Sigma$ at $M$, and so $C$ is an asymptotic curve on all surfaces $\Sigma$. Similar statements hold for the surfaces $\Sigma'$.

A final property may be mentioned. To each generator $MP$ of a surface $\Sigma_1$ of the family $\Sigma$ corresponds a generator $M'P'$ of a surface $\Sigma_1'$ of the family $\Sigma'$. These generators intersect at a point $R_1$ on the line $(\sigma, \sigma')$. But if one pair of generators of $\Sigma_1$ and $\Sigma_1'$ intersect then all do. Then $\Sigma_1$, $\Sigma_1'$ are tangent along a curve $C_1$, the locus of points $R_1$. The plane $MR_1M'$ is the osculating plane of $C_1$, and $C_1$ is an asymptotic curve on $\Sigma_1$ and $\Sigma_1'$. The two families of surfaces $\Sigma$, $\Sigma'$ may be separated into $\infty^1$ pairs of surfaces which are tangent in pairs along $\infty^1$ curves $C_\alpha$. The osculating planes of $C_\alpha$ form a pencil with $MM'$ as axis. The method used is that of moving frames of reference of Cartan.

*V. G. Grove* (East Lansing, Mich.).

**Akivis, M. A. Pairs of $T$-complexes.** Doklady Akad. Nauk SSSR (N.S.) **61**, 181–184 (1948). (Russian)

The purpose of this paper is to introduce the concept of $T$-pairs of complexes analogous to $T$-pairs of congruences due to S. P. Finikov. This is done by using the concept of harmonic intersection due to É. Cartan [Acad. Roum. Bull.

Sect. Sci. 14, 167–174 (1931)]. The author first shows that the $T$ property for a pair of congruences is equivalent to the following: there exists a pencil of demi-quadrics (that is, what Koenigs called one of the sets of generators of a ruled quadric) each of which intersects harmonically both congruences. Then he defines in an analogous way $T$-pairs of complexes. In the (Pluecker) five-dimensional interpretation, to a $T$-pair of complexes corresponds a focal congruence, that is, a 3-parameter family of rays such that every ray carries three foci and is contained in three developables. The author considers also an involutory system of complexes introduced by A. M. Vassiliev; if another system of complexes can be mapped on it so that the corresponding complexes form $T$-pairs then the second system is also involutory. This situation is also interpreted in five-space. *G. Y. Rainich* (Ann Arbor, Mich.).

**Gheorghiev, Gh. Surfaces dont les courbes des familles remarquables sont semblables.** Ann. Sci. Univ. Jassy. Sect. I. 30 (1944–1947), 75–140 (1948).

The study taken up in this paper is carried out by vector methods. The paper is composed of nine chapters. The chapter headings and some results follow. (I) Preliminaries to the kinematics of similarity transformations. Formulas for the study are set up. They are based on the similarity transformation $R=r+e^kr_1$, where $r$ is the position vector of the center of similitude, $e^k$ is the modulus. The notions of instantaneous centers and axes are introduced. Applications are made to the movements of a plane, of a line and of a sphere. (II) Applications to the transformations of curves and of homothetic satellite curves. General formulas are established for the correspondence between two curves, and applications are made to transformations for which directions intrinsic to these transformations are related in specific manners to the tangent, principal normal or binormal of the curves. There are thus set up some special similitudes, and some special families of curves called homothetic satellites. (III) Elements of a surface having a family of similar curves. The fundamental coefficients of the first and second kind are derived for such surfaces. (IV) Some resulting theorems. Applications of the results of the previous chapter are made to surfaces having similar geodesics, surfaces having similar asymptotic curves, and surfaces having similar lines of curvature. (V) Surfaces with similar asymptotics. Spiral surfaces having this property are discussed. (VI) Surfaces with similar geodesics. The equivalence is established of the problem of finding geodesics on spirals and that of the determination of curves determined by the torsion and one of its rectangular coordinates as functions of arc length. (VII) Surfaces with similar lines of curvature. A study is made of many special cases of such surfaces, and descriptions are made of various modes of generation of such surfaces. (VIII) Surfaces having similar cylindrical curves. It is shown that certain of such surfaces form part of a family of surfaces studied by A. Myller [Acad. Roum. Bull. Sect. Sci. 27, 103–104 (1944); these Rev. 9, 464]. (IX) On nautiloids. These surfaces belong to a family generated as follows. Let $C$ be a plane curve and $A$ a point on $C$. Through $A$ draw a plane $\pi$ normal to the plane of $C$. In $\pi$ there is a curve $\Gamma$ submitting to a similitude having $A$ as its center. The locus of $\Gamma$ is the surface in question. In particular if $C$ is a logarithmic spiral, $\pi$ the plane through $A$ and the pole of the spiral, the modulus of the similitude being proportional to the radius vector of $A$, then the locus of $\Gamma$ is the so-called nautiloid. [See Haton de la Goupillière, Ann. Sci. Acad. Polytech. Porto 3, 5–46, 69–204 (1908).]

*V. G. Grove* (East Lansing, Mich.).

**Gheorghiu, Gh. Th. Sur un couple des surfaces. I.** Bull. Sci. Tech. Polytech. Timişoara 13, 141–155 (1948).

Starting with the usual differential equations ($P$) (in asymptotic coordinates) for $x_{uu}$ and $x_{vv}$ of a surface in a projective three-space, the corresponding system ($A$) in an affine space may be easily deduced from ($P$) (requiring that $t=1$ is a solution of ($P$)). Remembering the role of the origin in a centro-affine geometry one completes ($A$) to a system ($C$-$A$) by expressing $x_{uv}$ as a linear combination of $x$, $x_u$, $x_v$. The author uses ($P$) in order to investigate a surface $S'$ projectively equivalent to a given surface $S$ and having (among others) the property that the asymptotic tangents $xx_u$ and $xx_v$ meet the corresponding tangents $x'x_u'$ and $x'x_v'$. They are three types of such surfaces $S_i$ ($i=0, 1, 2$). They may be described in the following way. Let $P$ ($\pi$) be a given fixed point (plane), $S$ a given surface and $Q$ a quadric which has contact of order two with $S$ (at the generator point of $S$) and which has $P$ and $\pi$ for conjugate elements. If $Q$ is the Lie quadric $L$ of the generator point, then $S=S_0$. If $Q$ and $L$ have (the asymptotic directions and) a degenerate conic section $C$ in common then $S=S_1$. If $C$ is not degenerate and its plane contains $P$ then $S=S_2$. Hence these surfaces belong to the centro-affine geometry. Using the device mentioned above one gets ($C$-$A$) from ($P$), which enables a detailed study of their different properties. For instance, they are characterized by $\Delta_1 \log G=\Delta_2 \log G$, where $\Delta_1$ and $\Delta_2$ are the differential operators with respect to the quadratic tensor which defines the asymptotic lines and $G$ is a function taken from the equation which expresses $x_{uv}$ as a linear combination of $x$, $x_u$, $x_v$. *V. Hlavatý.*

**Maeda, Kazuhiko. Differential geometries of right conoids in a normal-net under the linear isometric and generalized linear isometric transformation groups.** Sci. Rep. Tôhoku Imp. Univ., Ser. 1. 31, 41–50 (1942).

Let $a_0$ be a straight line, and ($Z$) the set of lines $g$ meeting $a_0$ at right angles. The set ($Z$) is called a normal-net of lines. Let $g_0$ be a fixed line of ($Z$) and $g$ an arbitrary line of ($Z$), making an angle of $\varphi$ with $g_0$ and at a distance of $\zeta$ from it. The quantity $z$ is defined by $z=\tan(\varphi+j\zeta)$ ($j^2=0$) is called the dual coordinate of the line $g$ with reference to $g_0$. A cylinder $\Sigma$ of radius $\frac{1}{2}$ is constructed with axis parallel to $a_0$ and intersecting the line of ($Z$) normal to both $g_0$ and $a_0$. An ellipse $E$ is constructed on $\Sigma$ and $P$ is a point of $E$. The lines of ($Z$) through $P$ generate a cylindroid $G$ and the projection of $P$ from a point on $g_0$ at a unit distance from $a_0$ on to the plane of ($a_0$, $g_0$) generates a parabola in that plane. The quantities $\theta=2\varphi$, $p=2\zeta$ defined in terms of the line $g$ defined by $\varphi$ and $\zeta$ are used to define a line in a plane $\pi$. So at a line ($p$, $\theta$) of $\pi$ there corresponds a line $g(\varphi, \zeta)$ of ($Z$), and to $g$ the point $P$ on $\Sigma$, and to $P$ its projection onto the plane ($a_0$, $g_0$). To $G$ in $\pi$ corresponds a circle. Let now $T_0$ be a collineation leaving $\Sigma$ invariant. Then $T_0$ induces on $\Sigma$ a transformation $T$ transforming the ellipse $E$ into an ellipse on $\Sigma$. To $T$ corresponds a transformation $T_1$ of ($Z$) which transforms a cylindroid into a cylindroid. And also to $T$ corresponds in $\pi$ a transformation $T_2$ of circles into circles, as well as a transformation $T_3$ transforming the parabola corresponding to $E$ in the plane ($a_0$, $g_0$) into a parabola. These transformations $T$, $T_1$, $T_2$, $T_3$

may be represented in the form

$$(*)\qquad z' = M_\lambda[(az+b)/(cz+d)], \qquad ad-bc\neq 0,$$
$$M_\lambda = \Re(z)+j\lambda\mathfrak{D}(z),$$

$\Re(z)$ and $\mathfrak{D}(z)$ being the real and dual parts of $z$. The transformation (*) is called a generalized linear isometric transformation. Such transformations form a group called the generalized linear isometric group.

An equation $z=z(t)$, $t$ being real, represents a right conoid $R$ of $(Z)$, a curve $C$ on $\Sigma$, or a curve $\Gamma$ in $(a_0g_0)$, or a curve $K$ in $\pi$. Let $R_1$, $R_2$ be two such conoids having $g$ in common. To $R_1$, $R_2$ correspond curves $C_1$, $C_2$ on $\Sigma$, $\Gamma_1$, $\Gamma_2$ in $(a_0g_0)$ and $(K_1, K_2)$ in $\pi$ having a tangent $\tau$ in common. The tangential distance $\mu$ along $\tau$ between $K_1$, $K_2$ is called the measure of $R_1$, $R_2$. A generalized isometric transformation preserving $\mu$ is called a linear isometric transformation. These transformations form the linear isometric group. These have the form $z'=(az+b)(cz+d)$ or $z'=(a\bar{z}+b)(c\bar{z}+d)$, $\Re(ad-bc)\neq 0$. The remainder of the paper is concerned with a study of the right conoid under the linear isometric group. A differential invariant $K$, called the curvature, and an integral invariant $\lambda$ are found. A right conoid with positive curvature $K$ is a helicoid or its transform, while the conoid for which $K=0$ is the linear isometric transform of an equilateral hyperbolic paraboloid. A differential and integral invariant of $R$ is found for the generalized linear isometric group. *V. G. Grove* (East Lansing, Mich.).

**Bol, Gerrit. Zur Projektivgeometrie der Flächenstreifen.** Arch. Math. 1, 192–199 (1948).

At every point $\mathfrak{x}(t)$ of a curve $C$ a plane $\mathfrak{X}(t)$ is given which is incident with $\mathfrak{x}$ and $\mathfrak{t}=\mathfrak{x}'$: $\mathfrak{X}\mathfrak{x}=\mathfrak{X}\mathfrak{x}'=\mathfrak{X}'\mathfrak{x}=0$. The factor of proportionality may be chosen in such a way that (1) $\mathfrak{X}'\mathfrak{x}'=-1$, $\mathfrak{X}'\mathfrak{x}''=0$. The generator point $\mathfrak{z}$ of the edge of regression of $\mathfrak{X}(t)$ together with $\mathfrak{x}$, $\mathfrak{t}=\mathfrak{x}'$ and $\mathfrak{y}=\mathfrak{x}''-a\mathfrak{x}$ $(2a=\mathfrak{X}''\mathfrak{x}'')$ constitutes the frame simplex for $C$ while the dual simplex consists of the planes $\mathfrak{X}$, $\mathfrak{T}=\mathfrak{X}'$, $\mathfrak{Y}=\mathfrak{X}''-a\mathfrak{X}$ and the osculating plane $\mathfrak{Z}$ of $C$. The corresponding "Frenet formulae" are

$$\mathfrak{x}'=\mathfrak{t},\quad \mathfrak{t}'=\mathfrak{y}+a\mathfrak{x},\quad \mathfrak{y}'=g\mathfrak{z}+a\mathfrak{t}+h\mathfrak{x},\quad \mathfrak{z}'=\bar{g}\mathfrak{x},$$
$$\mathfrak{X}'=\mathfrak{T},\quad \mathfrak{T}'=\mathfrak{Y}+a\mathfrak{X},\quad \mathfrak{Y}'=\bar{g}\mathfrak{Z}+a\mathfrak{T}-h\mathfrak{X},\quad \mathfrak{Z}'=g\mathfrak{X}.$$

Hence $g=0$ $(\bar{g}=0)$ means that $C$ is a plane curve (that the envelope of $\mathfrak{X}$ is a cone). The conditions (1) are preserved under $t\to t^*$ for $\mathfrak{x}^*=\varphi\mathfrak{x}$, $\mathfrak{X}^*=\varphi\mathfrak{X}$ $(\varphi=dt^*/dt)$. The point $\mathfrak{y}^*$ describes for $t=t_0$ the "osculating" conic section $\omega$ and the plane $\mathfrak{Y}^*$ describes the "osculating" cone which contains $\omega$. (If $C$ is a plane curve then $\omega$ reduces to the ordinary osculating conic.) Any quadric which has contact of order 4 with $C$ and whose polar plane with respect to $\mathfrak{z}$ contains the line $(\mathfrak{x}, \mathfrak{t})$ intersects the osculating plane $\mathfrak{Z}$ in $\omega$. The quadric which has with $\mathfrak{x}, \mathfrak{X}$ contact of order 3 (osculating quadric) is not in general the limiting position of a quadric which has 4 different points in common with $C$. Using these results the author states some theorems about the locus of $\omega$ as well as about the coefficients of the "Frenet formulae." For example, if and only if $h=0$, $(\bar{g}/g)'=0$, then $C$ is on a quadric $Q$ and $\mathfrak{X}$ are tangential planes of $Q$; if and only if $h=0=\bar{g}$, then $Q$ is a cone, and so on. *V. Hlavatý.*

**Stakowski, Walter. Invariantentheorie der Raumkurven im vierdimensionalen projektiven Raum.** Arch. Math. 1, 200–204 (1948).

Let $C(\mathfrak{x}(t))$ be a curve in a projective four space. The determinant $D=(\mathfrak{x}, \mathfrak{x}', \mathfrak{x}'', \mathfrak{x}''', \mathfrak{x}'''')$ is preserved by a transformation $\mathfrak{x}^*=(dt^*/dt)^2\mathfrak{x}$ (which combines a change of a factor of proportionality with a parameter transformation $t^*=t^*(t)$). Introducing these transformations, one may suppose, without loss of generality, $D=K$, where $K$ is a constant. The author supposes $K\neq 0$. The fundamental equation (belonging to $t$) which expresses $\mathfrak{x}^{\mathrm{v}}$ as a linear combination of $\mathfrak{x}$, $\mathfrak{x}'$, $\mathfrak{x}''$, $\mathfrak{x}'''$, may also be obtained from the linear differential equations for the "frame simplex"

$$(1)\qquad \mathfrak{x}'=\mathfrak{t},\quad \mathfrak{t}'=\mathfrak{y}+4a\mathfrak{x},\quad \mathfrak{y}'=\mathfrak{z}+6at-2b\mathfrak{x},$$
$$\mathfrak{z}'=\mathfrak{p}+6a\mathfrak{y}-3bt+\mu\mathfrak{x},\quad \mathfrak{p}'=4a\mathfrak{z}-2b\mathfrak{y}+\mu\mathfrak{t}+c\mathfrak{x}.$$

Denoting by $p_0, \cdots, p_4$ the local coordinates of a generator point of $C$ with respect to this frame (at $t=0$) one gets by means of (1) the expansions for $p_0, \cdots, p_4$ and

$$\Omega=4p_0p_4-4p_1p_3+2p_2^2=\frac{280}{9!}ct^9+\cdots.$$

Hence $\Omega=0$ is a quadric which has at least a nine point contact at $t=0$ with $C$. If $b=c=0$ then $C$ lies on it. [For a general theory of curves in a projective curved $n$-dimensional space cf. the reviewer's paper, Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] **2–3**, 119–144 (1935).] *V. Hlavatý* (Bloomington, Ind.).

**Colmez, Jean. Systèmes triples orthogonaux paratingents.** C. R. Acad. Sci. Paris **226**, 2043–2045 (1948).

G. Llensa has previously [cf. Bull. Sci. Math. (2) **65**, 225–250 (1941); same C. R. **220**, 297–298 (1945); **222**, 263–265 (1946); these Rev. **7**, 77, 173, 481] studied the level surfaces of a paratingent integral of the partial differential equation (E): $(\overline{MO}^2-R^2)\cdot\mathrm{grad}^2\,u=1$, where $M$ is the point $(x, y, z)$, $O=O(u)$ the center and $R=R(u)$ the radius of a sphere $S(u)$, a continuous function of one parameter $u$. The equation (E) was introduced by Darboux in connection with the problem of imbedding a surface in a triply orthogonal system. The author investigates the equation (E′): $|\mathrm{grad}\,u|=F(M, u)$, where $F(M, u)$ is a function continuous in $u$ and possessing continuous second derivatives with respect to $x$, $y$ and $z$. He transposes to (E′) the complete integral of (E) with spherical level surfaces used by Llensa and can extend to (E′) certain results of the latter. In addition, he gives a partial answer to a question of Bouligand [Bull. Math. Soc. Roumaine Sci. **35**, 57–67 (1933)] about a generalized form of Dupin's theorem. *C. Y. Pauc* (Cape Town).

**Chen, Yu Why. Branch points, poles and planar points of minimal surfaces in $R^3$.** Ann. of Math. (2) **49**, 790–806 (1948).

The author considers minimal surfaces $S$ defined on a closed Riemann surface $F$ of genus $g$. The surfaces are given by three differentials $d\varphi_j=(\partial x_j/\partial u-i\partial x_j/\partial v)(du+idv)$, $j=1, 2, 3$, which are assumed to be analytic up to $n$ poles on $F$. Then branch points are either the common zeros of the $d\varphi_j$ or are poles of higher order; as the author shows, the normal to $S$ can be defined continuously even at the poles and branch points. The author studies first the case when $S$ is uniquely defined by the differentials over $F$. He studies the nature of the surface in the neighborhood of a branch point and the relations between the genus $g$ of $F$, the number of branch points $b$ and the number of poles $p$ of $S$, and the index $t$ of the Gauss spherical mapping of $S$. A typical result of the latter type is the following equality: $b-2p=(2g-2)-2t$. To establish the latter result the author introduces a new differential on $F$ whose zeros and poles are the common zeros and poles of the $d\varphi_j$. This new differential

is used for several other purposes, for example, to count the number of branch points of minimal surfaces bounded by a finite number of smooth curves, and in particular, for minimal surfaces having polygonal boundaries. In the latter case, the author obtains some results that have also been derived by Courant. The author applies his results to a minimal surface studied by Stessmann [Math. Z. **38**, 417–442 (1934)]. *M. O. Reade* (Ann Arbor, Mich.).

**Mikan, Milan. La géométrie réglée non-euclidienne.** Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **46** (1945), 71–72 (1946).

Summary of the Czech original which appeared in Rozpravy II. Trídy České Akad. **55**, no. 6 (1945); these Rev. **9**, 619.

**Kuiper, N. H. On differentiable linesystems of one dual variable. II.** Nederl. Akad. Wetensch., Proc. **51**, 1244–1250 = Indagationes Math. **10**, 388–394 (1948).

The author applies his previous results [same vol., 1137–1145 = Indagationes Math. **10**, 361–369 (1948); these Rev. **10**, 326] to some special problems. Thus, for instance, the functions $P(T)$ and $Q(T)$ referred to in the first paper determine, but for a motion, a $D$-system. On the other hand an orthogonal reference system may be found for which Frenet formulae of a $D$-system are valid with a dual curvature $C$ and a dual torsion $T$. These equations admit different applications as for instance the construction of osculating $D$-systems. The last section is devoted to a formula in the line geometry which is similar to that of Euler-Savary on the sphere. *V. Hlavatý* (Bloomington, Ind.).

**Urban, Alois. Note on the T. Y. Thomas's paper: On the projective theory of two dimensional Riemann spaces.** Časopis Pěst. Mat. Fys. **73**, 89–92 (1948). (English. Czech summary)

In a two-dimensional projective space, T. Y. Thomas [Proc. Nat. Acad. Sci. U. S. A. **31**. 259–261 (1945); these Rev. **7**, 33] has introduced the vector density $gg^{ij}K_j$ $(i, j=1, 2)$ where $K_j$ is the derivative of the Gaussian curvature $K$. Such a space has a three-dimensional affine representation which is described by Thomas [The Differential Invariants of Generalized Spaces, Cambridge University Press, 1934, p. 55]. In this representation there is defined a "projective curvature tensor" ${}^*B^{\alpha}{}_{\beta\gamma\delta}$ $(\alpha, \beta, \gamma, \delta=0, 1, 2)$. The result of Urban's paper is the relation

$$gg^{ij}K_j=-\tfrac{1}{3}{}^*B^0{}_{rst}\epsilon^{ri}\epsilon^{st},$$

where we have used Thomas's notation and corrected a numerical slip of a factor of 2 in Urban's equation (19). This result also follows immediately from Thomas's book [see above], equation (18.7). *C. B. Allendoerfer.*

**Yano, Kentaro. Union curves and subpaths.** Math. Japonicae **1**, 51–59 (1948).

Im Riemannschen Raum $V_n$ sei in den Punkten einer Hyperfläche $V_{n-1}$ ein Vektorfeld $L^\lambda$ definiert. Diejenigen Kurven der $V_{n-1}$, deren Schmiegebene in Bezug auf den $V_n$ in jedem ihrer Punkte stets die dort vorgeschriebene $L^\lambda$ enthalten, werden als "union curves" bezeichnet. Verf. gibt das Differentialgleichungssystem zweiter Ordnung an, welches diese Kurven bestimmt. Längs einer beliebigen Kurve bestimmen diese Differentialausdrücke einen kontravarianten Vektor, den "union Krümmungsvektor" der Kurve, der offensichtlich eine Verallgemeinerung des Krümmungsvektors ist. Bekanntlich kann man, die durch die Übertragung vom $V_n$ im $V_{n-1}$ induzierte Übertragung dadurch erhalten, dass man das für den $V_n$ gebildete invariante Differential eines Vektors des $V_{n-1}$ in einem bestimmten Punkt, orthogonal auf die Tangentialhyperebene dieses Punktes des $V_{n-1}$ projiziert. Verf. projiziert nun das invariante Differential in Richtung $L^\lambda$ und erhält so eine Übertragung, die er als induzierte Übertragung bezüglich der Richtung $L^\lambda$ bezeichnet. Es wird nun nachgewiesen, dass die Bahnen (paths) in Bezug auf diese Übertragung die "union-Kurven" geben. Verf. definiert diese Kurvenklasse auch für einen affinzusammenhängenden Raum $A_n$. Zunächst speziell so, dass er die Richtung $L^\lambda$ zur Affinnormalen $B^\lambda$ der Hyperfläche $A_{n-1}$ macht. Die "union-Kurven" sind die Bahnen in Bezug auf die, auf der Hyperfläche induzierten Übertragung. Eine Verallgemeinerung erhält Verf. dadurch, dass er die $L^\lambda$ von der Affinnormale verschieden wählt, doch so, dass $L^\lambda$ den $A_{n-1}$ berührt. Es wird nun gefordert, dass die Schmiegebenen in Bezug auf den $A_n$ stets die durch $L^\lambda$, $B^\lambda$ bestimmte Ebene schneiden. Diese Kurven des $A_{n-1}$ bezeichnet Verf. als "subpaths." Verf. definiert nun auch "subpaths" im $A_n$ selbst dadurch, dass die Schmiegebene einer solchen Kurve in jedem ihrer Punkte stets den Vektor eines im $A_n$ erklärten Feldes enthalten soll. Verf. definiert als subprojektive Übertragung diejenigen Abänderungen der Übertragungsparameter der affinzusammenhängenden Mannigfaltigkeit, die die "subpaths" in sich überführen. Die subprojektive Geometrie ist dann der Inbegriff derjenigen Eigenschaften und Beziehungen, die bei subprojektiven Änderungen invariant bleiben. *O. Varga* (Debrecen).

**Yano, Kentaro. On the flat projective differential geometry.** Jap. J. Math. **19**, 385–440 (1947).

This is a purely expository paper, a comparative albeit unifying study of the various approaches to projective differential geometry (projective connection type). The four separate modes of development initiated by É. Cartan [1924], T. Y. Thomas [1926], O. Veblen [1929], and D. van Dantzig [1932] are presented in outline together with their interrelations and essential equivalence. This is supplemented by chapters on the various parametrizations of the paths, the differential equations of conics, and the affine hypersurface representation of projective spaces, material which stems from J. H. C. Whitehead, H. Hombu, the author himself, and others. All sources are dutifully acknowledged. Only the flat case is considered here. The author promises the general case later.
*J. L. Vanderslice* (College Park, Md.).

**Rozenfel'd, B. A. Conformal differential geometry of families of $C_m$ in $C_n$.** Mat. Sbornik N.S. **23**(65), 297–313 (1948). (Russian)

The paper presents a broad application of metric methods to the study of conformal differential geometry of $m$-spheres and $m$-planes in metric $n$-spaces. By $C_n$ is understood an $n$-space with Euclidean metric or a spherical $n$-space in a Euclidean $(n+1)$-space, a $C_n$ with an elliptic metric. A conformal $m$-sphere is either an $m$-sphere or an $m$-plane in $C_n$. The fundamental group of $C_n$ is the group of conformal transformations (the Möbius group) preserving contact and angle between hyperspheres. The author uses "$(n+2)$-spherical" coordinates chosen so that for a hypersphere $s\cdot s=(s^0)^2+(s^1)^2+\cdots+(s^n)^2-(s^{-1})^2=1$. Then the absolute has the equation $s\cdot s=0$. Thus a hyperbolic metric may be

introduced in $S_{n+1}$ so that the distance $\omega$ between two points is given by $\cos\omega = s\cdot t$. By observing that the group of hyperbolic motions in $S_{n+1}$ is simply isomorphic to the conformal group of transformations in $C_n$ and that the totality of hyperspheres of $C_n$ passing through a given $m$-sphere is represented in $S_{n+1}$ by an elliptic $(n-m-1)$-plane, it follows that the conformal differential geometry of $m$-spheres in $C_n$ is the same as the metric differential geometry of $(n-m-1)$-planes in a hyperbolic space $S_{n+1}$. Another observation enables the author to obtain conformal invariants for this geometry: the group of conformal transformations is a non-compact simple Lie group; it therefore has a Cartan metric in terms of which the space of $m$-spheres $C_n$ becomes a symmetric pseudo-Riemannian space. Thus one can study $m$-spirals, i.e., families of $m$-spheres corresponding to geodesics, and also congruences of $m$-spheres.

*M. S. Knebelman* (Pullman, Wash.).

**Ohkubo, Takeo. Über die Extensorrechnung in den verallgemeinerten Räumen von Flächenelementen höherer Ordnung.** J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 11, 1–37 (1946).

Der Begriff des Linienelementes höherer Ordnung, der den der erweiterten Punkttransformationen nach sich zieht, hat auch eine Erweiterung des Tensorbegriffs notwendig gemacht. Eine solche Verallgemeinerung wurde von H. V. Craig gegeben. Die Tensoren, die bei einer solchen Transformation erklärt sind, werden als Extensoren bezeichnet. Nach dem die Mannigfaltigkeit, die dem Aufbau von differentialgeometrischen Räumen zu Grunde liegt, von Cartan zunächst zur Linienelement bzw. Hyperflächenelement-Mannigfaltigkeit erweitert wurde, hat A. Kawaguchi Räume untersucht, deren Elemente Flächenelemente einer beliebigen Dimension seien können. Ist $x^i$ eine $N$-dimensionale Mannigfaltigkeit und ist in derselben durch $x^i = x^i(u^1, \cdots, u^K)$ eine $K$-dimensionale Fläche gegeben, so bestimmen die partiellen Ableitungen von $x^i$ bis zur $M$-ten Ordnung einschliesslich mit $x^i$ zusammen ein Flächenelement $M$-ter Ordnung. Der Inbegriff dieser Elemente bildet die Mannigfaltigkeit $M_N(M; K)$. Die Mannigfaltigkeit ist parametrisch, d.h. die Parameter dürfen nicht geändert werden. Bei einer Punkttransformation der $x^i$, erhält man für die Bestimmungsstücke des Flächenelementes höherer Ordnung ein Transformationsgesetz, das zusammen mit der Transformation der $x^i$ die erweiterte Koordinatentransformation bildet. In Bezug auf diese Transformation gebildete Tensoren, werden als verallgemeinerte Extensoren bezeichnet. Nachdem Verf. eine Reihe von algebraischen Betrachtungen über Extensoren anstellt, so z.B. die Darstellung von Exvektoren durch gewisse gewöhnliche Vektoren, wird der Raum durch Einführung eines Fundamentalextensors metrisiert. In dem metrischen Raum wird eine Übertragung und der Prozess der exkovarianten Differentiation erklärt. Die Überlegungen werden dann auf den Spezialfall einer, dem euklidischen Raum eingebetteten Untermannigfaltigkeit angewandt.

*O. Varga* (Debrecen).

## NUMERICAL AND GRAPHICAL METHODS

**✶Willers, Fr. A. Practical Analysis. Graphical and Numerical Methods. Translated by Robert T. Beyer.** Dover Publications, Inc., New York, 1948. x+422 pp. $6.00.

The German original was published in 1928 by de Gruyter, Berlin. The sections dealing with the slide rule and calculating machines have been rewritten to describe the equivalent instruments of American design.

**Murphy, R. B. Non-parametric tolerance limits.** Ann. Math. Statistics 19, 581–589 (1948).

The author's summary is as follows. In this note are presented graphs of minimum probable population coverage by sample blocks determined by the order statistics of a sample from a population with a continuous but unknown cumulative distribution function. The graphs are constructed for the three tolerance levels .90, .95, and .99. The number, $m$, of blocks excluded from the tolerance region runs as follows: $m=1(1)6(2)10(5)30(10)60(20)100$, and the sample size, $n$, runs from $m$ to 500. Thus the curves show the solution, $\beta$, of the equation $1-\alpha = I_\beta(n-m+1, m)$ for $\alpha = .90, .95, .99$ over the range of $n$ and $m$ given above, where $I_x(p, q)$ is Pearson's notation for the incomplete beta function. Examples are cited for the one- and two-variate cases. Finally, the exact and approximate formulae used in computations for these graphs are given.

*J. C. P. Miller* (London).

**Hillman, Abraham, and Salzer, Herbert E. The inverse functions of $z=w^{-1}\tanh w$ and $z=w^{-1}\coth w$.** J. Math. Physics 27, 202–209 (1948).

This paper discusses the theory of the functions

$$z=w^{-1}\tanh(w+k)$$

with $k=0$ or $k=\frac{1}{2}i\pi$ with a view to the evaluation of $w$ as a function of $z$. Expansions in the forms $w=\sum_0^\infty a_n z^n$ and $w=\sum_0^\infty b_n z^{-n}$ are developed as far as $n=10$ in literal form, and numerical values of the coefficients are given to from 7 to 10 decimals or figures (sometimes fewer for larger) for the first four branches; 10 coefficients are normally given (15 in two cases). Numerical values for the first five singularities are given in each case. [Care is needed in reading this paper, owing to excessive "economy" of notation.]

*J. C. P. Miller* (London).

**Orcutt, Guy H. A new regression analyser.** J. Roy. Statist. Soc. Ser. A. 111, 54–70 (1948).

This machine operates by analogue methods on input data expressed in digital form. The series to be analyzed are recorded by punching holes in cards. A feature of the machine is that each series occupies a separate card and time lags among two or more series can be introduced by mechanically shifting the relative positions of the cards. In order to perform the basic calculations required in regression analysis each digital series is converted to a time varying voltage. Electrical meter circuits can then be used to measure the sum of the items in a series, or the sum of the squares of the items, or the sum of the absolute values of the items. Constant voltage can be added to or subtracted from the individual items, and potentiometers can be used to introduce constant coefficients. The sum of the cross-products of two series is obtained by a watt-meter measurement, while a cross-product series itself can be produced by a double modulation method. Weighted means are obtained from adding circuits. A resistance-condenser circuit is used to obtain the mean of a series and the time-varying difference between the mean of a series and its instantaneous value. A number of examples are given to show how these basic procedures are applied in specific

problems and the design of an experimental machine is described. *S. H. Caldwell* (Cambridge, Mass.).

**Bodewig, E. L'approximation des racines complexes d'une équation transcendante à une inconnue.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 218–223 (1947).

This paper describes a graphical process for the arbitrarily close approximation of the complex roots of a transcendental equation in one unknown. There is also described a rapidly convergent method for the computation of the roots if one has a satisfactory first approximation. The latter method was given by Euler [Opera Omnia, ser. 1, v. 10, pp. 422–455]. *E. Frank* (Chicago, Ill.).

**Cassina, Ugo. Sul numero delle operazioni elementari necessarie per la risoluzione dei sistemi di equazioni lineari.** Boll. Un. Mat. Ital. (3) 3, 142–147 (1948).

The author counts the number of operations (additions and multiplications) in several methods for solving linear algebraic equations: the methods of elimination, that of Banachiewicz and that of the determinants by Chiò's rule, and finds that the two first involve exactly the same number, while the last one involves many more operations. However, Banachiewicz's method has many fewer recordings than Gauss's elimination method. *E. Bodewig.*

**Turing, A. M. Rounding-off errors in matrix processes.** Quart. J. Mech. Appl. Math. 1, 287–308 (1948).

This condensed paper consists of two parts: a discussion of methods for solving linear algebraic equations and a discussion of the errors to be expected. In the first part the author states that methods of approximation are more laborious than direct ones. Concerning the direct methods he shows that most of them are applications of the decomposition of the matrix $A$ into the product $LDU$, where $L$ and $U$ are triangular matrices with diagonal elements 1 and having elements not equal to zero below or above the diagonal, respectively, and $D$ is a diagonal matrix. He proves that the decomposition is unique. Applications of this are Gauss's elimination method with the Jordan variant which the author considers the best method for inverting $A$, Choleski's method, Morris's escalator process and Schmidt's orthogonalization. For each of these methods the author measures the work of computation by counting the number of multiplications and recordings. [It seems to the reviewer that the author's statement, that in solving several systems with the same $A$ it would be more convenient to invert $A$, is false since it is not necessary to compute the triangular matrix again in Gauss's method, for instance. On the contrary, Gauss's method is, from the author's point of view, the most economical one since every new system with the same $A$ requires only $n^2$ new multiplications.]

The second part begins by giving three measures of the magnitude of a matrix: the norm $N(A)$, the maximum expansion (von Neumann's "upper bound") and the maximum coefficient $M(A)$. The degree of ill-conditioning of a matrix is measured by the $N$-number $n^{-1}N(A)N(A^{-1})$ or by the $M$-number $nM(A)M(A^{-1})$. Orthogonal matrices have $N$-numbers 1 and thus are the best conditioned ones. The various rounding errors for Jordan's, Gauss's and Choleski's method are given. [In the reviewer's opinion most such determinations are impractical. It is more convenient to insert the solution into the given equations (which must be done in any case and requires only $n^2$ multiplications) and to determine the exact correction by solving a new system which will require only $n^2$ more multiplications.] *E. Bodewig* (The Hague).

**Hampl, Miloslav. Summation of series involving orthogonal functions applied to the solution of some technical problems.** Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 46 (1945), 69–70 (1946).

Summary of the Czech original which appeared in Rozpravy II. Třídy České Akad. 55, no. 5 (1945); these Rev. 9, 383.

**Lattmann, Max. Neue technische Mittel zur Behandlung mathematischer Probleme.** Mitt. Verein. Schweiz. Versich.-Math. 48, 19–36 (3 plates) (1948).

Expository lecture.

**Hartree, D. R., Newman, M. H. A., Wilkes, M. V., Williams, F. C., Wilkinson, J. H., and Booth, A. D. A discussion on computing machines.** Proc. Roy. Soc. London. Ser. A. 195, 265–287 (2 plates) (1948).

The respective contributions to the discussion are entitled: A historical survey of digital computing machines; General principles of the design of all-purpose computing machines; The design of a practical high-speed computing machine: the EDSAC; A cathode-ray tube digit store; The automatic computing engine at the National Physical Laboratory; Recent computer projects.

**Akušskiĭ, I. Ya. An "extremal" problem in the application of selectors to calculating-analytical machines.** Uspehi Matem. Nauk (N.S.) 2, no. 4(20), 183–186 (1947). (Russian)

The problem mentioned in the title is that of trying to accomplish as much as possible with a punch card tabulator having only a finite number of class selectors. The discussion is only general. *D. H. Lehmer.*

**Karpov, K. A. On the numerical solution of certain problems of analysis on tabulators with vertical-horizontal action.** Doklady Akad. Nauk SSSR (N.S.) 62, 741–744 (1948). (Russian)

A brief discussion is given of the use of a tabulator of "type Astra" as a difference engine. Values of a polynomial are built up by successive summation of differences in the usual way. One more step gives one the "integral" of the polynomial. *D. H. Lehmer* (Berkeley, Calif.).

**Sadovskiĭ, L. E. The algebraization of a problem in the theory of the control of calculating automata.** Uspehi Matem. Nauk (N.S.) 2, no. 6(22), 223–226 (1947). (Russian)

The author considers the problem of determining the minimum number of class selectors (as used on an I. B. M. tabulator, for instance) necessary in order that $n$ impulses (or sets of impulses) can be put on $n$ counters of the machine in any of the $n!$ possible ways by proper picking up of the selectors. Since for two pulses this can be effected by a pair of such selectors, the author shows that (for single pulses) the number sought is not greater than $2\nu$, where $\nu$ is the least number such that an arbitrary permutation $S$ of the symmetric group on $n$ letters can be expressed in at least one way in the form (*) $S=T_1^{e_1}T_2^{e_2}\cdots T_\nu^{e_\nu}$. Here the $e_i$ are 0 or 1 and the $T_i$ form a fixed ordered set of transpositions. The author shows that $\nu\leqq\frac{1}{2}n(n-1)$; and that in-

equality holds for $n=4$ and $n=5$. He suggests it would be an interesting problem to find a systematic method of determining $\nu$, and the form (*), for each $n$.
*H. B. Curry* (State College, Pa.).

**Rose, H. E. The mechanical differential analyser: its principles, development, and applications.** Inst. Mech. Engrs. Proc. **159**, 46–54 (4 plates) (1948).

**Fréchet, Maurice. Sur les expressions analytiques de la mortalité valables pour la vie entière.** J. Soc. Statist. Paris **88**, 261–285 (1947).

Most of the formulae used to graduate mortality tables are valid only for a certain range of ages. The author is interested in finding a formula which is applicable to the whole span of life. He proposes to fit a polynomial of degree six to $\log q_x$, taking as the independent variable $\log (x+2)-1.096$. The number 1.096 has no particular significance, but is introduced to simplify the numerical work. The polynomial is fitted by the method of least squares. *E. Lukacs* (China Lake, Calif.).

***Engelfriet, J. Une Théorie Générale de Récurrence en Matière d'Assurance sur la Vie et Contre l'Invalidité.** Martinus Nijhoff, The Hague, 1947. ii+78 pp. 3.60 guilders.

[Reprinted from Verzekerings-Arch. **27**, 1–78 (1947); these Rev. **9**, 211.] The author discusses a general form of insurance by assuming that a policy may be in any of a number of states. Suitable determinations of the states lead to the various conventional types of policies. Payments under the contract may be due as long as a certain state is maintained or upon transition from one state to another. Different states causing payments may coexist; return to a previous state is possible. This scheme is somewhat more general but essentially similar to the one designed by A. Loewy [S.-B. Heidelberger Akad. Wiss. Abt. A. **1917**, no. 6]. The greater generality causes the author to use a rather complicated set of symbols. *E. Lukacs.*

## ASTRONOMY

**Belorizky, David. Nouvelle méthode de calcul des éphémérides et des corrections des éléments des étoiles doubles.** C. R. Acad. Sci. Paris **227**, 893–895 (1948).

The method of ephemeris calculation of double stars proposed in this paper employs a rectangular coordinate system of which the origin is located at the central star and the $x$-axis is directed toward the projection of the periastron in the apparent orbit. The rectangular apparent coordinates are then expressed by simple formulae in terms of the projections of the major and minor axes in the apparent orbit, the angle between these axes, the eccentricity and the eccentric anomaly. The orientational elements of the true orbit are not required. Formulae for differential correction of the elements, expressed by these quantities, are also derived. *D. Brouwer.*

**Antunez de Mayolo, Santiago. Loi des forces dans un système gravitationnel du type soleil-planète.** Pont. Acad. Sci. Acta **4**, 89–94 (1940).

This paper is a slight extension of results of García [Revista Ci., Lima **45**, 219–280 (1944) = Actas Acad. Ci. Lima **7**, 163–224 (1944); these Rev. **6**, 75].
*W. Kaplan* (Ann Arbor, Mich.).

**Milne, E. A. Star-streaming and the stability of spiral orbits in spiral nebulae. I. Motion round a point-nucleus.** Monthly Not. Roy. Astr. Soc. **108**, 309–315 (1948).

Using the $t$-scale of time (i.e., one in which the galaxies are receding from one another), the author applied in two recent papers [same Not. **106**, 180–199 (1946); Astrophys. J. **106**, 137–142 (1947); these Rev. **8**, 607; **9**, 212] his kinematic theory of gravitation to obtain the equation of an arm of a spiral nebula as that of orbits which the mass particles are describing around its nucleus. In the present paper the author has investigated the question of stability of such orbits by Lindblad's method (using a rotating frame to which to refer the varied motion) and is led to conclude that, on the kinematic theory of gravitation, the spiral orbits around a galactic nucleus are stable and that small variations from ideal spirals should give rise to the phenomenon of star streaming exactly as in Lindblad's theory (which is based on the classical theory of gravitation). The velocity ellipse relative to a nonrotating frame is calculated in terms of the velocity ellipse relative to a rotating frame, and it is shown that the two axes of the ellipse interchange when we pass from the latter ellipse to the former, exactly as on the classical theory of gravitation. *Z. Kopal.*

**Milne, E. A. Star-streaming and the stability of spiral orbits in spiral nebulae. II. Motion in an extended distribution of matter.** Monthly Not. Roy. Astr. Soc. **108**, 316–323 (1948).

[Cf. the preceding review.] The aim of the present paper is a generalization, on the kinematic theory of gravitation, of the orbits of mass particles around a galactic nucleus to the case when the total mass of the galaxy is not wholly stored in a point-nucleus and the acceleration due to the material outside the nucleus must be considered. An appropriate expression (depending on time) is found for the resulting field of force and its consequences are compared with well-known results of Lindblad and Oort based on the classical theory of gravitation. It is shown, in particular, that known results in the theory of star streaming hold good on the classical and kinematic theories of gravitation alike.
*Z. Kopal* (Cambridge, Mass.).

**Lemaître, G. Modèles mécaniques d'amas de nébuleuses.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 551–565 (1948).

The author is concerned with the problem posed by the existence of large random velocities for the individual members of clusters of galaxies. Is it possible to account for the existence of more or less permanent concentrations of galaxies in which no single galaxy remains long in the same place? The two-fold purpose of the paper is to delineate the underlying mechanical model and to write down the fundamental equations of the problem. It is shown how these equations can be applied toward the solution of the well-known problem of uniform distribution in an homogeneous, expanding universe. As a preliminary to the study of the nonstatic solutions, two models in equilibrium are investigated, the first, an isotropic model in which the root-mean-square peculiar velocity is independent of the constant of

areas, the second, one in which an infinitely small concentration of a Gaussian character is assumed to exist.
*B. J. Bok* (Cambridge, Mass.).

**Kiewiet de Jonge, Joost H. On the relationships between the frequency functions of stellar velocities.** Proc. Nat. Acad. Sci. U. S. A. **34**, 553–561 (1948).

In the past most studies on the relations between the frequency functions of radial velocities have started by assuming that certain analytical interpolation formulae would represent the observed data. The present paper presupposes no such analytical formulae, but makes use of direct numerical representation of the observations. The author makes use of a new type of average frequency function, obtained by averaging, with equal weights, of the frequency functions for standard areas of equal size, thus eliminating all effects of galactic concentration. It is shown that these functions are related by simple relations, well-adapted to numerical computation. The author indicates that the present paper represents a study preliminary to an investigation of the distribution of absolute magnitudes of stars of given spectral type on the basis of their known radial velocities and proper motions. *B. J. Bok.*

**Miczaika, G. R. Bemerkungen zur hydrodynamischen Behandlung von Sternsystemen.** Astr. Nachr. **276**, 169–172 (1948).

The paper opens with a brief discussion of two definitions of the mean free path according to Chandrasekhar, the first referring to energy transfer in encounters, the second to deflections resulting from encounters. In our galactic system the mean free paths may vary between $10^5$ and $10^9$ parsecs, with $10^6$ parsecs being a plausible value for the vicinity of the sun. The author stresses that in distances of the order of 1% to 10% of the mean free paths, stars experience considerable changes in their kinetic energies. A discussion of the value of the Reynolds number shows that turbulence is to be expected in the central part of most galactic systems, but that laminar flow may occur in the outer parts.

It is shown that, as a consequence of the free paths' being comparable to the dimensions of the system, the concepts of kinetic gas theory are not applicable to stellar systems. The statistical-hydrodynamical equations of motion are reduced to the Navier-Stokes equations. The paper concludes with some critical observations concerning galactic rotation interpreted as a consequence of general streaming under the limiting assumptions of stationary conditions and negligible pressure friction terms. *B. J. Bok* (Cambridge, Mass.).

**Kourganoff, Vladimir. Sur l'application pratique de la méthode variationnelle au calcul de modèles d'atmosphères stellaires.** C. R. Acad. Sci. Paris **228**, 300–302 (1949).

The minimal principle described by the author [same C. R. **225**, 1124–1126 (1947); these Rev. **9**, 310] for solving the equation of transfer for conservative isotropic scattering is extended to the equation of transfer appropriate for the radiative equilibrium of a stellar atmosphere in local thermodynamic equilibrium and with a continuous absorption coefficient varying with wavelength. In this latter problem it is the integrated flux, $\int_0^\infty F_\nu d\nu$, which is constant. The author therefore suggests that in the practical solution we may replace the integral over $\nu$ by a sum taken over some "strategically" chosen value of $\nu$. *S. Chandrasekhar.*

**Pal, G., and Bandyopadhyay, G. Note on homologous and adiabatic radial motion of a star.** Bull. Calcutta Math. Soc. **40**, 64–68 (1948).

This paper examines certain conditions which appear necessary for homologous radial motion of a star under adiabatic conditions. Homologous radial motion appears possible in a star with any initial relation between pressure and density, provided the adiabatic exponent has the value 4/3. Certain initial conditions are required simultaneously. It is further shown that only a homogenous star can undergo homologous radial motion if the adiabatic condition is different from 4/3. *G. Randers* (Oslo).

**Bandyopadhyay, G. On slow homologous contraction of stars.** Proc. Nat. Inst. Sci. India **14**, 29–43 (1948).

The possibility of slow homologous contraction of stellar configurations is investigated. The object of the work is simply to obtain relations restricting the laws of opacity and energy generation in a homologously contracting star. Some such properties are found. The Cowling model appears not to admit of homologous contraction. *G. Randers.*

# RELATIVITY

**Kowalewski, Gérard. Képler et les formules de Lorentz.** C. R. Acad. Sci. Paris **227**, 762–763 (1948).

If $v$, $u$ are the true and eccentric anomalies for an elliptic orbit of eccentricity $\epsilon$, and if $x=\cos u$, $y=\sin u$, $x=\cos v$, $y=\sin v$, then

$$(*)\qquad x_1=(x-\epsilon)/(1-\epsilon x),\quad y_1=y(1-\epsilon^2)^{\frac{1}{2}}/(1-\epsilon x).$$

For $\epsilon$ varying from 0 to 1, these equations define a group of projective transformations for which the author suggests the name Keplerian group. The product of the transformations $\epsilon_1$, $\epsilon_2$ is the transformation $\epsilon_3$, where $\epsilon_3=(\epsilon_1+\epsilon_2)/(1+\epsilon_1\epsilon_2)$, and this, with $\epsilon_\nu=v_\nu/c$, gives the law of addition of velocities in special relativity. Moreover, if formulae (*) are written in the homogeneous form $x_1=(x-\epsilon z)/(1-\epsilon^2)^{\frac{1}{2}}$, $y_1=y$, $z_1=(-\epsilon x+z)/(1-\epsilon^2)^{\frac{1}{2}}$, then the first and last of these, with $z=ct$, $\epsilon=v/c$, become the Lorentz equations $x_1=(x-vt)/(1-v^2/c^2)^{\frac{1}{2}}$, etc. Thus, remarks the author, Kepler had these celebrated equations in his hands.
*H. S. Ruse* (Leeds).

**Müller, Hans Robert. Zyklographische Betrachtung der Kinematik der speziellen Relativitätstheorie.** Monatsh. Math. **52**, 337–353 (1948).

In this article the "cyclographic" method of representation in descriptive geometry developed by E. Müller and J. L. Krames [Müller, Vorlesungen über darstellende Geometrie, v. 2, Deuticke, Wien, 1929] is applied to Minkowski's treatment of special relativity. It is claimed that, although no new physical results emerge, the descriptive geometer will welcome this new line of approach. The Lorentz transformation, time dilation, Fitzgerald contraction, and Einstein's velocity addition theorem are discussed in turn with

the aid of elaborate geometrical constructions. The author concludes with some remarks on the role of the rigid body in relativistic mechanics. *G. J. Whitrow* (London).

**Kar, S. C. Die Lorentztransformation und ihr physikalischer Inhalt.** Bull. Calcutta Math. Soc. **40**, 83–106 (1948).

Distance and time interval between events are defined by parallax measurements. Given an event, the space-time coordinates obtained by two observers are shown to be related by a Lorentz transformation. *A. Schild.*

**Buchdahl, H. A. On Eddington's higher order equations of the gravitational field.** Proc. Edinburgh Math. Soc. (2) **8**, 89–94 (1948).

The author considers gravitational field equations obtained from the following Lagrangian densities: (i) $(-g)^{\frac{1}{2}}R_{\mu\nu}R^{\mu\nu}$, (ii) $(-g)^{\frac{1}{2}}R^2$, (iii) $(-g)^{\frac{1}{2}}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. It is shown that every empty space solution of Einstein's field equations (with cosmological term) is also a solution of the equations obtained from (i) and (ii); every space of constant curvature satisfying Einstein's equations, and the Schwarzschild line element, are also solutions of the equations obtained from (iii). As in Weyl's theory, the dimension 4 of space-time enters the arguments. *A. Schild* (Toronto, Ont.).

**Ludwig, Günther. Skalares Materiefeld in der projektiven Relativitätstheorie mit variabler Gravitationsinvarianten.** Z. Naturforschung **2a**, 482–489 (1947).

Equations for a scalar field variable are considered which represent neutral or charged particles. The equations are solved under certain specialized conditions in order to study the influence of a variable "constant" of gravitation. The energy-momentum tensor is computed and is used to illustrate some general properties. It is shown that the world line of charged particles are geodesics of a 5-dimensional space. *A. Schild* (Toronto, Ont.).

**Ludwig, Günther, und Müller, Claus. Ein Modell des Kosmos und der Sternentstehung.** Ann. Physik (6) **2**, 76–84 (1948).

An expanding closed universe is considered. Field equations obtained from projective relativity are applied to this model and they are shown to imply a "constant" of gravitation which decreases with time. Two limiting cases are discussed in which the pressure is (a) zero, and (b) $\frac{1}{3}$ energy density. In either case certain dimensionless arbitrary constants appear which the authors assume to be of the order of magnitude 1. Case (a) is then shown to lead to cosmological relations obtained previously by Jordan, and case (b) corresponds to Jordan's theory of the origin of stars. *A. Schild* (Toronto, Ont.).

**Ludwig, Günther, und Müller, Claus Ernst Friedrich. Ein Modell des Kosmos und der Sternentstehung.** Arch. Math. **1**, 80–82 (1948).

A shortened form of the preceding paper. *A. Schild* (Toronto, Ont.).

**Jordan, P. Der Zusammenhang der vier- und fünfdimensionalen Metrik.** Ann. Physik (6) **3**, 153–155 (1948).

A simple derivation is given for relations between the four-dimensional metric $g_{kl}$ and the five-dimensional metric $g_{\mu\nu}$ of projective relativity theory. *H. C. Corben.*

**Bergmann, Peter G. Non-linear field theories.** Physical Rev. (2) **75**, 680–685 (1949).

The formal properties of classical covariant field equations are examined and the form of the conservation laws and of the equations of motion are derived without recourse at this stage to the introduction of a metric. The nature of the field variables is left general, and a Lagrangian principle involving the first partial derivatives of the field variables with respect to the coordinates is postulated. Canonical momenta are introduced and the uniqueness of the solutions is discussed, and it is shown that the infinitesimal contact transformation leading from the state at time $t$ to that at $t+dt$ contains four arbitrary functions of the space coordinates. In the corresponding quantized theory which it is proposed to develop this will lead to auxiliary initial conditions on the state vector which are automatically preserved as time increases. It is hoped that quantization of this general nonlinear field theory will be able to exploit the great advantage of general relativity theory in which the equations of motion are contained in the field equations themselves, so that field singularities may be treated without the introduction of infinite interaction terms. *H. C. Corben* (Pittsburgh, Pa.).

**Tonnelat, Marie-Antoinette. Théorie unitaire du champ physique. I. Les tenseurs fondamentaux et la connexion affine.** C. R. Acad. Sci. Paris **228**, 368–370 (1949).

The author outlines a program for determining a non-symmetrical affine connection from a variational principle. *A. H. Taub* (Urbana, Ill.).

## MECHANICS

**Maeda, Kazuhiko. A proof of Euler-Savary's formula.** Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **31**, 51–54 (1942).

A proof, by means of complex numbers, of the Euler-Savary formula in plane kinematics. The author considers also the more general case of the curvature of the envelope of a curve in the moving plane. *O. Bottema* (Delft).

**Rosenauer, N. On the construction of velocities of kinematic chains and mechanisms.** Contrib. Baltic Univ. no. **19**, 5 pp. (1947).

The known theorem that, for a carried line, the projections of the velocities of different points of the line on the line are equal enables a graphical process which is applied to the derivation of the velocities of linkages employing one or two ternary links. (A ternary link is one which is joined to three other links at three distinct points.) The paper is essentially the same as an earlier paper by the author in German [Z. Angew. Math. Mech. **17**, 173–176 (1937)]. *M. Goldberg* (Washington, D. C.).

**Rosenauers, N. On the construction of accelerations of kinematic chains and mechanisms including slide couples in movable planes.** Contrib. Baltic Univ. no. **38**, 14 pp. (1947).

The following theorem is demonstrated and applied to several linkage mechanisms which have slide couples in movable members. If in a chain or mechanism two links are connected through a slide couple, the representations of two points form a parallelogram, two sides of which are the relative normal accelerations of these points and the other

two are the Coriolis accelerations. The extremities of the different accelerations of these points form a second parallelogram with sides which are mutually perpendicular to the sides of the first. Two of these sides are the slide accelerations and the other two are relative tangential accelerations of these points. The construction is simpler than one given by Beyer in "Technische Kinematik" [Barth, Leipzig, 1931]. *M. Goldberg* (Washington, D. C.).

**Rosenauers, N. On the construction of accelerations of kinematic chains and mechanisms.** Contrib. Baltic Univ. no. **32**, 10 pp. (1947).

The acceleration of a point $A$ in a link is equal to the acceleration of the point $B$ in the same link plus the acceleration of $A$ around $B$. This equation is resolved into its tangent and normal components in accordance with graphical methods described by Grübler and Wittenbauer. These methods are applied to the linkages of the foregoing review. Related methods have been described and referenced by Federhofer [Graphische Kinematik und Kinetostatik, Springer, Berlin, 1932, pp. 20–22]. *M. Goldberg.*

**Čerkudinov, S. A. The method of best approximation in the synthesis of mechanisms.** Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1517–1530 (1948). (Russian)

The problem proposed is to approximate a curve (1) $q(x, y)=0$, where $q$ is a polynomial, by the trajectory $x=x(z)$, $y=y(z)$, $x$ and $y$ polynomials in $z$, of a point $D$ of the connecting rod of a four-bar linkage. If the curve (1) is embedded into a family $Q(x, y; t)=0$, where $Q$ is a polynomial in $x, y$ and $Q(x, y; 0)=q(x, y)$, the equation (2) $g(z, t)=Q(x(z), y(z); t)=0$ arises, where $g$ is a polynomial in $z$ and depends also on the dimensions of the linkage. If the latter are so chosen that equation (2) is satisfied "as nearly as possible," the author regards the problem as solved. The definition of the best possible approximate compliance with the equation $g(z, t)=0$, for $z_0\leqq z\leqq z_m$ and $|t|\leqq L$, claimed to be superior to that of Čebyšev, is the following. (a) The extrema of $t$ shall all have the same absolute value $L$. (b) The number $m$ of the extrema shall be a maximum for all the possible dimensions of the linkage. (c) At the endpoints of the interval $(z_0, z_m)$ the $t$ values are $-L$ and $L$.

The determination of the "best approximation" proceeds by successive attempts to make $m$ equal to $n, n-1, \cdots$. If $m=n$ is attempted, $z_1, \cdots, z_{m-1}$ must exist satisfying

$$g(z, L)=(z-z_1)^2(z-z_3)^2\cdots(z-z_{m-2})^2(z-z_m)$$

and

$$g(z, -L)=(z-z_0)(z-z_2)^2\cdots(z-z_{m-1})^2.$$

Familiar formulas then establish a set of expressions of the values of the coefficients of $g$ at $t=\pm L$ as polynomials in $z_0, \cdots, z_m$. Since these values depend on the linkage parameters, elimination of the $z$'s results in a set of equations for these parameters. If these equations are not compatible, $m=n-1$ is tried, and so forth. Examples are given and applications to Watt's linkage and crankshaft curves presented with numerical illustrations. *A. W. Wundheiler.*

**Bautin, N. N. On the motion of an idealized model of clocks with two degrees of freedom. (A model of the Galileo-Huyghens clocks.)** Doklady Akad. Nauk SSSR (N.S.) **61**, 17–20 (1948). (Russian)

Andronov and Neumark [C. R. (Doklady) Acad. Sci. URSS (N.S.) **50**, 17–20 (1946); these Rev. **8**, 101] proposed a clock model preserving the basic features of a clock as a system with two degrees of freedom and at the same time greatly simplifying computations. However, they only applied their scheme to clocks of pre-Galilean type, i.e., without pendulum or spring. The author applies the same method to clocks of modern type, i.e., with a pendulum or a spring. A procedure for obtaining the periodic motion is indicated and stability conditions of rather general but intricate nature are given in the paper. *S. Lefschetz.*

**Reeb, Georges. Sur les mouvements périodiques de certains systèmes mécaniques.** C. R. Acad. Sci. Paris **227**, 1331–1332 (1948).

Let $V_n$ be the configuration space of a holonomic dynamical system subject to constraints independent of time $t$, to forces derived from a potential $U$ and to periodic forces of period $\lambda$ which dissipate energy, for sufficiently large velocities. It is assumed that $V_n$ is a compact Riemannian manifold; the space of tangent vectors to $V_n$ of norm not greater than $h$ is denoted by $\bar{W}_h$. It is then shown that the transformation $T_\lambda$, which maps the point $P_t$ of a trajectory on the point $P_{t+\lambda}$, can be interpreted as a continuous map of $\bar{W}_h$ into itself, for $h$ sufficiently large, and has an algebraic number of fixed points equal to the characteristic $\chi$ of $V_n$. Hence, if $\chi\neq 0$, the system admits periodic orbits.

*W. Kaplan* (Ann Arbor, Mich.).

## *Hydrodynamics, Aerodynamics*

**✶Streeter, Victor L. Fluid Dynamics.** McGraw-Hill Book Company, Inc., New York, 1948. xi+263 pp. $5.00.

This book is intended for use in a second course in fluid mechanics. After discussing briefly the basic concepts of fluids, including stress relationships, the author proceeds to ideal fluids (here defined as frictionless and incompressible). Most of the volume is devoted to this case, covering the following subjects: basic flow definitions and theorems for two- and three-dimensional flow; applications of complex variable, including Blasius' theorems, Schwarz-Christoffel transformation and free streamlines; and vortex motion. The three chapters on viscous fluids cover the development of the Navier-Stokes equations, applications and boundary-layer theory. Certain mathematical methods beyond elementary calculus are employed, but in each case these are introduced and developed for the benefit of readers with insufficient mathematical training.

The author states that he has drawn heavily upon Lamb's "Hydrodynamics," and this is clearly the case. Apparently his intention has been to provide a textbook for engineering students that would give them much of the basic content of that standard treatise. There are problems in each chapter. For the most part, these seem to be rather elementary, but there are exceptions, such as the problems on plane flow about obstacles, which will require more originality. The unwary reader might be warned that the term "dynamic pressure" is used in an unconventional connotation.

It seems unfortunate that the author, in his development of useful mathematical techniques, has not included contour integration, residue theory, or any mention of Laurent's theorem. He does not employ vector or tensor notation. Nevertheless, the presentation is clear and logical, and the volume should be successful in bringing a considerable body of applicable mathematics, as well as fluid mechanics, to science students. *W. R. Sears* (Ithaca, N. Y.).

**Cârstoiu, Ion. Sur certaines formules intégrales dans le mouvement d'un fluide.** C. R. Acad. Sci. Paris **227**, 1337–1339 (1948).

A formula is given for the volume described by the tip of the acceleration vector erected at every point of a closed surface in a perfect fluid. This formula is expressed in terms of the invariants of the quadric associated with the acceleration field in the fluid [see Acad. Roum. Bull. Sect. Sci. **29**, 207–214 (1946); these Rev. **10**, 72].

*W. J. Nemerever* (Ann Arbor, Mich.).

**Cope, W. F. The equations of hydrodynamics in a very general form.** Ministry of Aircraft Production, Aeronaut. Res. Committee, Rep. and Memoranda no. **1903** (6387), 6 pp. (1942).

Using vector notation, the author obtains the equations of motion of a fluid for which neither viscosity nor conductivity is necessarily constant. *J. L. Synge* (Dublin).

**Thiruvenkatachar, V. R. The analogue of Blasius' formula in subsonic compressible flow.** Proc. Nat. Inst. Sci. India **14**, 339–342 (1948).

The following is an extract from the author's introduction. The object of this note is to derive a formula for the force in subsonic compressible flow, which is the analogue of the well-known Blasius formulae in the incompressible case. The derivation is carried out on the basis of the hodograph method as recently developed by C. C. Lin. It is also shown that the familiar Prandtl-Glauert rule is derivable from the formula. *C. C. Lin* (Cambridge, Mass.).

**Jacob, Caius. De l'influence de la compressibilité sur les écoulements fluides.** Disquisit. Math. Phys. **6**, 193–223 (1948).

This is apparently another version of the author's previous work on subsonic gas jets [Acad. Roum. Bull. Sect. Sci. **28**, 637–641 (1946); these Rev. **9**, 543].

*Y. H. Kuo* (Ithaca, N. Y.).

**Castoldi, Luigi. Superficie e linee di Bernoulli nel moto stazionario di un fluido reale.** Atti Accad. Ligure **4** (1947), 21–25 (1948).

In extension of Bernoulli's theorem, Lamb [Proc. London Math. Soc. (1) **9**, 91–92 (1878)] proved that in any steady motion with velocity $V$, if an acceleration potential $W$ exists then it follows that the quantity $V^2/2+W$ has a constant value on each surface everywhere tangent to the vortex-lines and the stream-lines. The author attempts to formulate a corresponding theorem for barotropic motions of a viscous fluid of constant viscosity, subject to conservative extraneous force. Writing the dynamical equation in the form curl $V\times V+\nu$ curl curl $V=$grad $\Phi$, he proves that $\Phi=$constant on each vortex-line and stream-line if and only if $(\nu V\times$curl $V)\times$curl curl $V=0$. He gives examples to show that motions satisfying his conditions exist. The brackets in his formula (2) are a misprint. *C. Truesdell.*

**Angelitch, Tatomir. Sur l'application de la méthode de Pfaff dans la dynamique des fluides.** Acad. Serbe Sci. Publ. Inst. Math. **2**, 211–222 (1948). (French. Serbian summary)

The author shows that the theory of Pfaffian systems can be used to derive the dynamical equation for viscous fluids under the unnecessary and in general incorrect assumption that the motion is barotropic. *C. Truesdell.*

**Germain, Paul. Quelques remarques géométriques sur les équations aux dérivées partielles. Application à la dynamique des gaz.** C. R. Acad. Sci. Paris **228**, 163–165 (1949).

Let $x_i$ and $p_i$ $(i\leqq 3)$ be rectangular coordinates in Euclidean spaces $E$ and $H$ with parallel $x_i$ and $p_i$ axes. Consider the map $x\rightarrow p_i=\partial\varphi/\partial x_i$, where

$$(*)\qquad \sum A_{ij}(x,\varphi,p)\partial^2\varphi/\partial x_i\partial x_j+B(x,\varphi,p)=0.$$

Cases of special interest are (1) the map of a 3-dimensional region is a curve or surface; (2) the map $dp_i$, $\delta p_i$ of a surface element $dx_i$, $\delta x_i$ is indeterminate. Case (1) leads to special solutions of (*), (2) to the characteristic conditions. For steady 3-dimensional irrotational flow the partial differential equation is of the form (*) with $A_{ij}=A_{ij}(p)$, $B=0$. The spaces $E$ and $H$ are the physical and hodograph spaces. The characteristic conditions are useful for numerical calculation of 3-dimensional supersonic flows. Generalized Prandtl-Mayer flow, produced by deflecting a uniform supersonic stream over a developable surface, maps onto a curve. Swept-back plane flow maps onto a surface. Conical flow fields are counter-examples to theorem 3, which asserts that 2-dimensional maps are plane. *J. H. Giese.*

**Robinson, A. On source and vortex distributions in the linearized theory of steady supersonic flow.** Quart. J. Mech. Appl. Math. **1**, 408–432 (1948).

This paper was previously issued as Coll. Aeronaut. Cranfield. Rep. no. **9** (1947); these Rev. **10**, 74. Two figures and a "review of principal points" have been added.

*P. A. Lagerstrom* (Pasadena, Calif.).

**Guderley, G. Nonstationary gas flow in thin pipes of variable cross section.** Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1196**, 81 pp. (1948).

[Translation of Zentrale für wissentschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB), Forschungsber. no. **1744** (1942).] The author gives a straightforward treatment of the characteristic method for nonstationary flow in a pipe of slowly varying cross-section such that the flow is quasi-one-dimensional. The transformations of the compatibility equations are introduced through the simpler case of a perfect gas whose specific heat is a function of temperature only, flowing in a pipe of constant cross-section at constant entropy. The flow with variable section is considered next. Finally the general case of varying entropy and cross-section is treated. The author shows that, even for this complicated case, the separation of variables in the compatibility equations is still possible if the gas is perfect. The numerical iteration procedure in each case is the "lattice point" method and is discussed in detail. The paper concludes with a study on the transition conditions at the compression shocks and method of computing such shocks in the flow. Tables of quantities necessary for computation are included.

*H. S. Tsien* (Cambridge, Mass.).

**Schlichting, H. Die Grenzschicht mit Absaugung und Ausblasen.** Luftfahrtforschung **19**, 179–181 (1942).

Die Beeinflussung der Grenzschicht durch Ausblasen und Absaugen gewinnt neuerdings erhebliche Bedeutung. Für die ebene laminare Grenzschicht an einer längs angeströmten ebenen Platte mit Ausblasen und Absaugen an der Wand sollen im folgenden einige einfache Abschätzungen und für

einen Sonderfall der Absaugung eine einfache Lösung der Grenzschichtgleichungen mitgeteilt werden.
*Author's summary.*

**Schlichting, H. Ein Näherungsverfahren zur Berechnung der laminaren Reibungsschicht mit Absaugung.** Ing.-Arch. 16, 201–220 (1948).

The approximate calculation of the flow in an incompressible laminar boundary layer with boundary layer suction is undertaken using the integral form of the momentum equation and choosing an appropriate one-parameter family of velocity profiles. The distribution of velocity $u$ parallel to the surface is taken, when the free stream velocity is $U$, to be $u/U=F_1(\eta)+KF_2(\eta)$, where $\eta=y/\delta_1(x)$ and $K$ depends upon $x$ for a given problem. Here $F_1(\eta)$ is chosen to be the asymptotic solution for uniform boundary layer suction: $F_1(\eta)=1-e^{-\eta}$ [see the preceding review or Pretsch, Z. Angew. Math. Mech. **24**, 264–267 (1944); these Rev. **10**, 337], while $F_2(\eta)$ is chosen to give a good approximation to the Blasius profile for the flat plate without pressure gradient:

$$F_2(\eta)=\begin{cases}F_1(\eta)-\sin(\frac{1}{6}\pi\eta), & 0<\eta<3,\\ F_1(\eta)-1=-e^{-\eta}, & \eta\geqq 3.\end{cases}$$

The value of $K(x)$ is completely determined by the boundary conditions and the distribution of suction velocity at the surface. Consequently all features of the solution follow from the determination of the one parameter $Z=\vartheta^2/\nu$, where $\vartheta$ is the momentum thickness. The differential equation for $Z$ which results from the momentum integral equation may be solved by the isocline method for an arbitrary distribution of free stream and suction velocities. Charts of the parameters involved are provided to assist in the numerical process which is completely described.

Several examples illustrate the application and accuracy of the method. The examples of the flat plate, circular cylinder, and symmetric Joukowski profile at zero angle of attack, all with uniform suction, are presented in detail.
*F. E. Marble* (Pasadena, Calif.).

**Fage, E., et Vernet-Lozet, M. Une méthode de calcul des vitesses à la surface d'un profil d'aile, en écoulement plan d'un fluide parfait incompressible.** C. R. Acad. Sci. Paris **227**, 1339–1341 (1948).

In the transformation of an airfoil profile into a circle, one can compute the logarithm of the speed ratio for corresponding points, by means of a Poisson integral, in terms of the angle between the velocity vectors at such points. The difficulty is that the correspondence between points on the profile and on the circle is not known a priori. The author states that this correspondence does not vary much from profile to profile and can be considered known. This approximation obviously leads to great simplification of the process, but its accuracy seems somewhat doubtful.
*W. R. Sears* (Ithaca, N. Y.).

**Manwell, A. R. Aerofoils of maximum thickness ratio for a given maximum pressure coefficient.** Quart. J. Mech. Appl. Math. **1**, 365–375 (1948).

It is shown, by an argument using conformal mapping, that if an airfoil possesses the property described in the title (or maximum ratio of area to chord-squared), then it must be such that at every point the velocity equals the maximum, except for points at the extremes of the chord. This is proved for both incompressible and subsonic compressible flow, and is extended to axisymmetric bodies and flow involving symmetrically placed disturbances. For incompressible flow, the profile having this property was found by Riabouchinsky [Proc. London Math. Soc. (2) **19**, 206–215 (1921)]. In principle, the problem is also solved here for adiabatic or isothermal compressible flows, but an explicit solution can be derived only by means of the von Kármán-Tsien approximation [Tsien, J. Aeronaut. Sci. **6**, 399–407 (1939); these Rev. **2**, 168]. *W. R. Sears.*

**Kaplan, Carl. Effect of compressibility at high subsonic velocities on the lifting force acting on an elliptic cylinder.** Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **834**, 9 pp. (1946).

Previously published as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1118** (1946); these Rev. **8**, 107.

**Miles, John W. Harmonic and transient motion of a swept wing in supersonic flow.** J. Aeronaut. Sci. **15**, 343–346 (1948).

The results of previous papers [same J. **14**, 351–358 (1947); **15**, 565–568 (1948); these Rev. **8**, 610; **10**, 163; and one unpublished] are extended to the case of a yawed uniform wing whose leading and trailing edges are ahead of the Mach waves. Elementary solutions of the linearized potential equation for an oscillating source and line source are used. It is concluded that the nonsteady motion of this airfoil at Mach number $M$ is equivalent to that of an unyawed airfoil having the same chord and chordwise velocity distribution in the flight direction, if the forces on the latter are reduced by a factor $\cos\sigma$ and its Mach number is taken as $M\cos\sigma$, where $\sigma$ denotes the angle of yaw. The treatment of flutter of a swept wing is discussed in the light of this result. *W. R. Sears* (Ithaca, N. Y.).

**Miles, John W. Transient loading of airfoils at supersonic speeds.** J. Aeronaut. Sci. **15**, 592–598 (1948).

The pressure distribution on a two-dimensional thin airfoil carrying out sinusoidal oscillations (deformations) in supersonic flight was calculated, using linearized theory, in an earlier paper [same J. **14**, 351–358 (1947); these Rev. **8**, 610]. By expressing the boundary conditions for entry into a sharp gust and for sudden flap deflection in Fourier integrals, the author is able to calculate the pressure, lift, and moment for these cases. The "Wagner problem" of sudden change of incidence is included. The results are in closed form and are plotted against the distance travelled, for three Mach numbers: 1.2, $\sqrt{2}$, and 2, and a comparison is made with the incompressible case. *W. R. Sears.*

**Strang, W. J. A physical theory of supersonic aerofoils in unsteady flow.** Proc. Roy. Soc. London. Ser. A. **195**, 245–264 (1948).

The problems of entry into a sharp gust and sudden change of incidence (Wagner's problem) are attacked, using linearized theory, for a flat supersonic airfoil in plane flow. The method consists essentially of superimposing the effects of stationary line sources which begin to operate at the instant the leading edge reaches them. The pressure on the airfoil and finally the transient lift are calculated. This calculation has advantages of physical clarity. The reviewer finds the gust results to be in agreement with those of Miles [see the preceding review]. For an airfoil performing steady oscillations without angular velocity the lift is calculated by superposition using the solution to the Wagner case. The author states that the results are in complete

agreement with those of Temple and Jahn [Royal Aircraft Estab. Rep. no. **SME-3314** (1945)], Garrick and Rubinow [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1383** (1947); these Rev. **9**, 216], and others, all of which were obtained by a somewhat different approach. An approximate theory by Collar [Royal Aircraft Estab. Rep. no. **SME-3278** (1944)], is based on a false conjecture, and the reason is discussed here. Finally, the transient lift on loaded wings, free to move in vertical translation, is calculated for both sharp-edged and linearly graded gusts. *W. R. Sears.*

**Sauer, R. Überschallströmung um beliebig geformte Geschossspitzen unter kleinem Anstellwinkel.** Luftfahrtforschung **19**, 148–152 (1942).

Das Charakteristikenverfahren für achsensymmetrische Überschallströmungen wird auf nahezu achsensymmetrische Strömungen erweitert, wie sie bei Drehkörpern vorkommen, die unter kleinem Anstellwinkel angeblasen sind. Das Verfahren lässt sich auf beliebig geformte Geschossspitzen anwenden und vereinfacht sich erheblich bei kegeligen Spitzen. *Author's summary.*

**Lotkin, Mark M. Vorticity in the supersonic flow about yawing cones.** J. Aeronaut. Sci. **15**, 656–660 (1948).

The approximate calculation of Stone [see J. Math. Physics **27**, 67–81 (1948); these Rev. **9**, 544] for a cone yawed a small amount $\epsilon$ is reviewed. Stone neglected $O(\epsilon^2)$ but accounted for the nonuniform entropy behind the shock wave. Sauer [see the preceding review] neglected these, assuming irrotational flow. The present paper is intended to assess the influence of vorticity by comparing Stone's and Sauer's numerical results. It appears that the angle of yaw of the (nearly conical) shock wave is pretty well approximated but that the normal force on the cone may be seriously overestimated, by Sauer's theory.
*W. R. Sears* (Ithaca, N. Y.).

**Biot, M. A. Transonic drag of an accelerated body.** Quart. Appl. Math. **7**, 101–105 (1949).

The author calculates the rectilinear uniformly accelerated motion of an infinite wedge in perfect compressible fluid. By using the linearized differential equation for the potential and the technique of superposition of solutions, the pressure on the surface of the wedge can be calculated by direct quadrature. The pressure has an infinity at the nose of the wedge in contrast to the uniform pressure over the surface for constant supersonic motion. For the particular instant when the wedge moves with the sonic velocity $c$, the drag coefficient $c_D$ for a length $l$ of the wedge is $c_D=25.5\alpha^2(c^2/2\gamma l)^{\frac{1}{2}}$, where $\alpha$ is the small semi-wedge angle, $\gamma$ the constant acceleration. The author does not mention the fact that, by linearizing the differential equation, he limits himself to very large accelerations for transonic flow. This limitation of linear theory was pointed out by Lin, Reissner, and Tsien [J. Math. Physics **27**, 220–231 (1948); these Rev. **10**, 162]. *H. S. Tsien* (Cambridge, Mass.).

**Meksyn, D. Note on stability of laminar viscous flow between parallel planes.** Proc. Roy. Soc. London. Ser. A. 195, 174–179 (1948).

The author shows that the transformation of the slowly varying integrals, obtained from his treatment of the equation of hydrodynamic stability, can be identified with those given earlier by Heisenberg and Tollmein, although the results appear in different forms. [Reviewer's remark. For the existence proof of solutions having the usually considered asymptotic representations, see W. Wasow, Ann. of Math. (2) **49**, 852–871 (1948); these Rev. **10**, 377.]
*C. C. Lin* (Cambridge, Mass.).

**Kudryašev, L. I. A generalized integral relation for the thermal boundary layer and its application to the calculation of heat exchange.** Doklady Akad. Nauk SSSR (N.S.) **63**, 23–26 (1948). (Russian)

The author outlines an analytical determination of the heat transfer coefficient for incompressible turbulent boundary layer along a flat plate with arbitrary ambient temperature gradient in the direction of the flow along the plate. A generalized integral relation is developed analogous to that for momentum thickness growth. The principal analytical tool is a generalized temperature boundary layer thickness:

$$\delta=\int_0^\infty (u/u_0)(1-t^{k+1}/t_0^{k+1})dy.$$

The effect of the temperature gradient on the heat transfer is expressed as a multiplicative correction to the formula for constant ambient temperature: $Nu=.69(Pr)^{\frac{1}{2}}(Re)^{\frac{1}{2}}$.
*N. A. Hall* (Minneapolis, Minn.).

**von Kármán, Theodore. Progress in the statistical theory of turbulence.** Proc. Nat. Acad. Sci. U. S. A. **34**, 530–539 (1948).

After a concise review of the progress of the statistical theory of turbulence, from Reynolds to Kolmogoroff, the paper is devoted to an exposition of the results briefly given in a previous note [C. R. Acad. Sci. Paris **226**, 2108–2111 (1948); these Rev. **10**, 216]. Special attention is given to the case (case of large Reynolds numbers, neglecting the viscosity term in the equation of transfer of energy) in which the function $F(\kappa)$ preserves its shape during the decay; then it can be written in the form $F(\kappa)=(\overline{u^2}/\kappa_0)\Phi(\kappa/\kappa_0)$, where $\kappa_0$ is a function of time. An interpolation formula is suggested: $\Phi(\xi)=c\xi^4(1+\xi^2)^{-17/6}$, $c$=constant, which is supposed to represent $\Phi$ correctly for small and large values of $\xi$, and has the advantage that all calculations can be carried out analytically by use of known functions. Comparison of observed [Liepmann and Laufer] and computed values of the triple correlation $g(r)$ shows that the agreement is excellent for values of $g$ larger than 0.1, but after that the measured values are higher than the calculated ones. *J. Kampé de Fériet* (Lille).

**Kovasznay, Leslie S. G. Spectrum of locally isotropic turbulence.** J. Aeronaut. Sci. **15**, 745–753 (1948).

The author investigates a turbulent velocity field that is locally isotropic in Kolmogoroff's sense [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 301–305 (1941); **32**, 16–18 (1941); these Rev. **2**, 327; **3**, 221]; he considers the energy spectrum $E(n)$ ($n$, wave number) and the energy transport function $S(n)$ (amount of energy that passes from Fourier components that are of wave number less than $n$ to components of wave number greater than $n$); introducing the "Reynolds number of a Fourier component" $R(n)=(1/\nu)\{E(n)/n\}^{\frac{1}{2}}$, by dimensional considerations, he puts $S(n)=K[R(n)]E^{3/2}n^{5/2}$.

As working hypothesis the author assumes that, in locally isotropic turbulence, the function $K[R(n)]$ is a universal constant $K$; he thus gets the equation

$$\frac{dE}{dn}=-\frac{5}{3}\frac{E}{n}-\frac{8}{3}\frac{\nu}{R_0}\left\{\frac{E}{n}\right\}^{\frac{1}{2}},\qquad R_0=15\pi^2/K.$$

Neglecting viscosity, one gets the "$-5/3$ power law"; in the general case, one has for the spectrum

$$E(n)=E_0(n/n_0)^{-5/3}[1-(n/n_0)^{4/3}]^2,$$

where $E_0=\nu^2 n_0/R_0$. The values of the Reynolds number $R(n)$ computed by this formula are compared with measurements of Simmons; for values of $p=n^{-4/3}$ between 3 and 15 the agreement is fairly good; the discrepancies are in the same region as for Kolmogoroff's theory, at the low frequency end of the spectrum. *J. Kampé de Fériet* (Lille).

**Batchelor, G. K. Energy decay and self-preserving correlation functions in isotropic turbulence.** Quart. Appl. Math. 6, 97–116 (1948).

This paper contains a critical review and some extensions of several existing assumptions of self-preservation of the correlation functions in isotropic turbulence and the law of decay thus obtained. Emphasis is placed on the fact that a complete similarity of the correlation function can only exist and will exist at low Reynolds numbers. It is pointed out that the longitudinal correlation function $f(r,t)$ must then be of the form $e^{-r^2/8\nu t}$, where $\nu$ is the kinematic viscosity coefficient. This is shown to be the only solution of the Kármán-Howarth family with a Loitziansky invariant neither zero nor infinity. [Reviewer's remark. Another type of self-preservation has since been discussed by the reviewer in the paper reviewed below.] *C. C. Lin.*

**Lin, C. C. Note on the law of decay of isotropic turbulence.** Proc. Nat. Acad. Sci. U. S. A. 34, 540–543 (1948).

Putting the double and triple correlation functions $f(r)$ and $h(r)$ (with the usual notations for the study of isotropic turbulence) in the form

$$u'^2[1-f(r)]=v^2\beta_2(r/\eta),\quad u'^3h(r)=v^3\beta_3(r/\eta)$$

($\eta$ and $v$ being characteristic measures of length and velocity) the author makes the assumption of self-preservation of $\beta_2$ and $\beta_3$ during the process of decay. It follows that $\eta=(\nu^3/\epsilon)^{1/4}$, $v=(\nu\epsilon)^{1/4}$, $\epsilon=-\frac{3}{2}du'^2/dt$. Thus one gets for the law of decay of turbulence (at high Reynolds numbers): $\epsilon=C(t-t_0)^{-2}$, $u'^2=\alpha(t-t_0)^{-1}+\beta$, and for the change of Taylor's microscale $\lambda$: $\lambda^2=10\nu(t-t_0)[1+(\beta/\alpha)(t-t_0)]$, where $C$, $\alpha$, $\beta$ are constants. The author checks this result with experimental results of Batchelor and Townsend; the agreement is good. When $\beta=0$, one has the half-power law of decay and consequently a self-preservation of $f(r)$ itself; in any case, this is approximately satisfied for the initial stages of decay when $t-t_0$ is small. *J. Kampé de Fériet.*

**Djang, Gwoh-Fan. A kinetic theory of turbulence.** Chinese J. Phys. 7, 176–191 (1948).

In the kinetic theory of gases a number of statistical quantities called "state functions" are usually introduced: pressure, temperature, internal energy, entropy, etc.; the author tries to adapt this method of attack to the case of the turbulent flow of a fluid. He suggests defining a "turbulence temperature" by $T_R=\overline{u'^2}\lambda_m^2/\nu^2$, where $(\overline{u'^2})^{1/2}$ is, as usual, the square root mean of the velocity fluctuations at a point of the fluid, and $\lambda_m$ "the square root mean of different sizes of turbulence." [This $\lambda_m$ seems to the reviewer to need more explanation and closer investigation than is supposed in the paper.] Following to some extent Boussinesq, the author assumes that the turbulent stress has the following form:

$$-\rho\overline{u_j'u_k'}=\epsilon\left(\frac{\partial\bar{u}_j}{\partial x_k}+\frac{\partial\bar{u}_k}{\partial x_j}\right)-\tfrac{1}{3}\rho\delta_{j,k}\overline{u'^2},$$

$$\delta_{j,j}=1,\ \delta_{j,k}=0;$$

he supposes that the turbulent viscosity $\epsilon$ is related to the "turbulence temperature" $T_R$ in the same manner as the viscosity $\nu$ to the thermal temperature $T$; treating "turbulence as a gas consisting of particles of sizes $\lambda_m$," he puts $\epsilon=k\lambda_m\sqrt{T_R}$ ($k$=constant), thus reducing the determination of $\epsilon$ to the computation of $T_R$ and $\lambda_m$; for $T_R$ only, there is given a general equation, similar to the equation of transfer of heat. Three applications (flow through a circular pipe, flow between two infinite planes and decay of turbulence) are discussed; but lacking any general equation for $\lambda_m$, the author has to make, in each case, new "reasonable assumptions"; then the comparisons of his results with experiments are not of very great value to test his theory; they seem only to check his guess on the particular value of $\lambda_m$ in each case. *J. Kampé de Fériet* (Lille).

**Lenz, W. Schiffswellen in elementarer Behandlungsweise.** Ann. Physik (6) 1, 75–82 (1947).

The wave pattern created by a point source moving over the surface of water of infinite depth is treated by working with a superposition of plane waves rather than the more customary method of working with a superposition of cylindrical waves. The resulting integral representation for the surface waves is evaluated approximately by the method of stationary phase to yield the shape of the wave crests (but not the amplitude) for the case in which capillary forces are dominant as well as the case in which the force of gravity is dominant. The classical results of Kelvin and Havelock are obtained. *J. J. Stoker* (New York, N. Y.).

**Friedlander, F. G. On the total reflection of plane waves.** Quart. J. Mech. Appl. Math. 1, 376–384 (1948).

The author studies the reflection and refraction of transverse plane waves at an interface which is parallel to the direction of polarization. The case considered is that in which the incident wave is of arbitrary shape and the angle of incidence exceeds the critical angle. The solution of this problem depends on the solution of a functional equation which involves a desired harmonic function of two variables. This equation can be solved by using the half plane analogue of Poisson's formula. The same type of mathematical analysis may be applied to certain problems in electromagnetic theory as well as to the propagation of waves of arbitrary shape over the surface of a semi-infinite elastic solid. *A. E. Heins* (Pittsburgh, Pa.).

**Lighthill, M. J. The position of the shock-wave in certain aerodynamic problems.** Quart. J. Mech. Appl. Math. 1, 309–318 (1948).

The author tackles the problems associated with the motion and strength of shock waves in three types of one-dimensional flow: (i) motion caused by the uniform expansion of a cylinder into still air, (ii) motion caused by the uniform expansion of a sphere into still air, (iii) supersonic flow past a symmetrically placed cone. Since the ordinary linearized theory fails in predicting shock wave strength, the author proceeds as follows. He replaces the (ordinary) differential equation of motion by a soluble, still nonlinear, simplification having the same linearized form as the original. The quantity of interest is derived from the solution of this simpler equation, and a suitable approximation made.

Armed with this result of the rough analysis, he then derives the same approximation rigorously from the full equation. He finds that in both cases (i) and (ii) the shock wave strength vanishes as the velocity $v$ of the mechanical expansion goes to zero. But whereas in the former case the strength drops off as $v^4$, in the latter it goes to zero like $\exp[-(\gamma+1)^{-4}v^{-3}]$. In problem (iii) he applies the same method to the classical problem of supersonic flow past a cone and compares his results with those of linearized theory on the one hand, and the exact computations of Taylor and Maccoll on the other. *D. P. Ling* (Murray Hill, N. J.).

**Kromm, Alexander. Zur Ausbreitung der Stosswellen in Kreislochscheiben.** Z. Angew. Math. Mech. **28**, 297–303 (1948).

The author provides numerical data for his first paper [same vol., 104–114 (1948); these Rev. **9**, 636] and indicates how one arranges the formulae to get them.
*A. E. Heins* (Pittsburgh, Pa.).

**Ertel, Hans. Eine Methode zur approximativen Vorausberechnung von Luftmassenverlagerungen.** S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1948**, no. 3, 23 pp. (1948).

A system of displacement equations is developed whose solution shows the air mass arriving at a given spot at a future time. Since the equations and their solution take into account only the first-order terms of the Taylor series by which the quantities have to be expressed, the solution represents a first approximation. The air mass is assumed to move horizontally along the earth's surface. Even with this assumption the required terms involving the horizontal variations of the horizontal wind velocity components cannot be determined with sufficient accuracy in practice. Therefore a process of successive approximation is given which permits a practical solution and which converges provided that the forecast interval is chosen sufficiently short, in two examples half a day and a day, respectively, depending on the spatial variation of the wind field. In practice this successive approximation can be done graphically in a simple manner. *B. Haurwitz.*

**Maple, C. G., and Synge, J. L. Aerodynamic symmetry of projectiles.** Quart. Appl. Math. **6**, 345–366 (1949).

This paper deals with the effect of various types of rotational and reflectional symmetry on the aerodynamic force system acting on a projectile (but not with the dynamics of the projectile). The paper uses complex numbers and the appropriate cyclotomic group. The first few terms in assumed power series are studied. Rotational $n$-gonal symmetry and reflectional symmetry are separately treated. The cases of no axial spin and again of no cross spin yield further special results. Regions of stability are established in certain cases. Fourteen explicit diagrams show which terms (up through $D=10$) in the expansions of $(F, G)$ and $(F_3, G_3)$ survive under rotational symmetry, where $(F, G)$ are (complex) transverse force and couple components, while $(F_3, G_3)$ are similarly (real) axial components. The axial torque $G_3$ in the case of a nonspinning projectile with reflectional and $n$-gonal rotational symmetry ($n>2$), has no absolute, nor linear, term, and only two second-degree terms, together written as $k(u_3)(u\bar{\omega}+\bar{u}\omega)$, where $k$ is a real function, whose algebraic sign (determining stability) must, however, be found ordinarily by experiment alone.
*A. A. Bennett* (Providence, R. I.).

## *Elasticity, Plasticity*

**Rachkovitch, Daniel. Le potentiel d'un corps élastique sous forme diadique.** Acad. Serbe Sci. Publ. Inst. Math. **1**, 136–142 (1947).

**Rachkovitch, Daniel. Forme dyadique des équations fondamentales de la théorie d'élasticité.** Acad. Serbe Sci. Publ. Inst. Math. **2**, 248–256 (1948). (French. Serbian summary)

This paper contains textbook material on elasticity, developed in a dyadic notation. *C. Truesdell.*

**Ferrandon, Jean. Sur la caractère rotationnel de la déformation sans cavitation d'un continu élastique, déterminée par le petit déplacement d'un solide.** Houille Blanche **3**, 445–446 (1948).

The author notes that if a rigid inclusion embedded in an elastic continuum is displaced, the deformation is necessarily rotational. He had essentially proved this in a previous paper [C. R. Acad. Sci. Paris **226**, 2047–2048 (1948); these Rev. **10**, 167] by assuming the existence of a displacement potential and meeting a contradiction.
*G. F. Carrier* (Providence, R. I.).

**Cattaneo, Carlo. Teoria del contatto elastico in seconda approssimazione: compressione obliqua.** Rend. Sem. Fac. Sci. Univ. Cagliari **17** (1947), 13–28 (1948).

**Chilton, E. G. Large deformations of an elastic solid.** J. Appl. Mech. **15**, 362–368 (1948).

The stress-strain relationship due to Hencky [J. Rheology **2**, 169–176 (1937)] is considered. It concerns large elastic strains in isotropic homogeneous materials, and, expressed in terms of principal directions, gives a relationship between stress and logarithmic or natural strain. This law is applied to the specific cases of pure tension, compression, bending and torsion, and to "simple" shear parallel to a fixed plane (the author terms this "pure" shear contrary to the usual convention which reserves this term for the case of fixed principal axes). For large strains all quantities are expanded to the second power of the strain, and it is found that in the case of simple shear and torsion a secondary stress system is required in addition to the primary shear stress corresponding to the strain. For all types of loading the theory agrees with the measured relationships between force and deformation in natural rubber specimens. *E. H. Lee.*

**Hruban, K. The semi-infinite solid with variable modulus of elasticity.** Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **46** (1945), 151–166 (1946).

Simple radial stress solutions are obtained for the semi-infinite heterogeneous isotropic solid under point loading normal to the boundary and for the plane strain case with the force at any angle. Results are also given for load distributed over a circular area and a long strip. The method used is to solve the equations of elasticity by setting all stresses equal to zero except the radial. The variations of Young's modulus $E$ with the radial coordinate $r$ and polar angle $\varphi$ which permit particular displacement solutions are then determined. Poisson's ratio $1/m$ is taken as constant and the displacement solutions chosen are those which vanish as $r$ becomes infinite. Among the variations of $E$ for which answers are tabulated are $E_0(r\cos\varphi)^n$, $m=n+1$; $E_0\cos^k\varphi$, $m=2$, for the three-dimensional problem and $E_0(r\cos\varphi)^{\frac{1}{2}}$, $m=5/2$; $E_0 r$, $m=2$, for plane strain.
*D. C. Drucker* (Providence, R. I.).

**Sen, Bibhutibhusan. Direct determination of stresses from the stress equations in some two-dimensional problems of elasticity. V. Problems of curvilinear boundaries.** Philos. Mag. (7) **39**, 992–1001 (1948).

[For part IV, see the same Mag. (7) **36**, 66–72 (1945); these Rev. **7**, 41.] It is shown that the stress components for a state of generalized plane stress can be expressed in terms of $\Theta=\widehat{xx}+\widehat{yy}$, which is harmonic, and explicitly in terms of a pair of conjugate harmonic functions $\varphi$ and $\psi$. This formulation facilitates conversion into the orthogonal curvilinear system of coordinates $\xi+i\eta=f(x+iy)$. Appropriate choices of $\Theta$, $\varphi$ and $\psi$ are demonstrated to determine the stress distributions for a plate under uniform tensile stress at infinity, and containing a hole in the form of (1) the inverse of an ellipse, (2) the loop of a lemniscate, (3) an elliptic limaçon, and (4) an approximate square. In addition the problem of an elliptic hole in an infinite plate with uniform normal pressure on the elliptic boundary is solved by this method. *E. H. Lee* (Providence, R. I.).

**Bahšiyan, F. A. The rotation of a rough cylinder in a visco-plastic medium.** Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 749–756 (1948). (Russian)

The paper is concerned with the flow of a visco-plastic material (Bingham material) between two coaxial cylindrical surfaces, the outer one of which is at rest, while the angular velocity of the inner one is a given function of time. *W. Prager* (Providence, R. I.).

**Phillips, Aris. Calculation of the displacements in plastic torsion.** J. Math. Physics **27**, 270–273 (1949).

Studying the torsion problem for a prismatic bar made of an incompressible plastic material with strain-hardening, the author proposes to derive the equations for the displacements "by making a minimum of assumptions regarding the stress distribution." He then proceeds to stipulate the validity of a stress-strain law of the deformation type which furnishes a unique state of strain for any given state of stress. As A. A. Ilyushin has shown [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 207–218 (1945); these Rev. **7**, 144; translated in Plastic Deformation, Principles and Theories, pp. 97–116, Mapleton House, Brooklyn, N. Y., 1948; these Rev. **10**, 170] a stress-strain law of this mathematically convenient type will furnish correct results only if, for each material element, the principal axes of strain and the ratios of the principal strains remain constant throughout the deformation process. The author does not verify whether this condition is fulfilled in the torsion problem. In fact, it is fulfilled only in certain special cases where the direction of the shearing stress at any point of the cross section is determined by the geometry of the cross section independently of the stress-strain law. The example treated in the paper (infinite strip) is one of these exceptional cases, the shearing stress being everywhere parallel to the boundary of the strip. Another exceptional case is the circular cross section. In these special cases, however, the problem can be solved by more elementary methods than those used in the present paper. *W. Prager.*

**Borowicka, H. Die Druckausbreitung in einer Halbscheibe bei mit der Tiefe abnehmendem Elastizitätsmodul.** Österreich. Ing.-Arch. **2**, 360–363 (1948).

A simple solution in closed form is given for the plane stress distribution in an elastic half-plane due to a concentrated normal force on the boundary, assuming Young's modulus to be inversely proportional to the distance from the boundary. By integration the solution is found for a normal load uniformly distributed over a segment of the boundary. This follows the author's earlier paper [Ing.-Arch. **14**, 75–82 (1943); these Rev. **6**, 138]. *H. J. Greenberg* (Pittsburgh, Pa.).

**★Lehnickiĭ, S. G. Anizotropnye Plastinki. [Anisotropic Plates].** OGIZ, Moscow-Leningrad, 1947. 355 pp.

This is a new impression of a book [first published in 1944] concerned with a systematic treatment of the theory of thin anisotropic elastic plates subjected to small deformations. Its object is to acquaint individuals working with anisotropic media with the present state of development of a branch of elasticity that is assuming great technical importance. The book develops three basic themes: (1) the state of generalized plane stress in anisotropic media [chapters 1–7, 138 pp.], (2) small deflections of thin anisotropic plates [chapters 8–11, 138 pp.], (3) stability of thin anisotropic plates [chapters 12–15, 102 pp.]. The presentation is condensed and the stress is placed on the practical side of the theory in order to make the book serve as a guide to designers of structures built of anisotropic materials. Many conclusions are summarized in the form of graphs and tables. The theoretical results are often stated without proofs with the indication of sources where proofs may be found. Bibliographical references include 80 items, with the latest entry dated 1943. This is the first comprehensive exposition of anisotropic theory of elasticity in book form known to the reviewer. *I. S. Sokolnikoff* (Los Angeles, Calif.).

**Parasyuk, O. S. An elastic-plastic problem with a non-biharmonic plastic state.** Doklady Akad. Nauk SSSR (N.S.) **63**, 367–370 (1948). (Russian)

L. A. Galin [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 367–386 (1946); these Rev. **8**, 241] studied the plane elastic-plastic strain in an infinite plate with a circular hole when the state of stress at infinity is given and the hole is exposed to a given uniform pressure. As was pointed out by W. Prager [Rev. Math. Union Interbalkan. **2**, 45–51 (1938)] the stress function in the plastic zone surrounding the hole is biharmonic in this case, a condition which was essential for the success of Galin's approach to the problem. In the present paper the problem is complicated by the addition of a uniform shearing stress acting along the contour of the hole. The stress function in the plastic zone surrounding the hole is then no longer biharmonic. The author succeeds in adapting Mušelišvili's method in elasticity to the treatment of the problem. It is claimed that the method developed in this paper could be applied to numerous problems of plane elastic-plastic strain. *W. Prager* (Providence, R. I.).

**Bauer, F. Die dreiseitig gelagerte und am freien Rand belastete rechteckige Platte.** Österreich. Ing.-Arch. **3**, 1–8 (1949).

Explicit solutions are obtained for problems of bending of rectangular plates, when two opposite edges of the plate are simply supported and when of the two other edges one is simply supported or built-in and the other is acted upon by forces or moments. *E. Reissner.*

**Filonenko-Borodich, M. M. The bending of the rectangular plate with two clamped opposite edges.** Vestnik Moskov. Univ. **1947**, no. 3, 29–36 (1947). (Russian. English summary)

This is a sequel to the author's paper [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 193–208

(1946); these Rev. 7, 437] in which the system of "almost orthogonal functions" $P_n(x)=\cos n\pi x/a-\cos (n+2)\pi x/a$, $n=0, 1, 2, \cdots$, is applied to solve approximately the problem of deflection of a thin elastic rectangular plate clamped on the edges $x=0$, $x=a$, and with arbitrarily prescribed boundary conditions along $y=0$ and $y=b$. The normal load $g(x, y)$ is symmetric with respect to the line $x=a/2$. The approximate deflection $w$ is sought in the form $w=\sum_{m=0}^{n}P_{2m}f_{2m}(y)$, where the functions $f_{2m}(y)$ satisfy the system of fourth-order differential equations arising from the deflection equation $\nabla^4 w=g(x, y)/D$. The solution for a rectangular plate clamped along all edges is indicated.
*I. S. Sokolnikoff* (Los Angeles, Calif.).

**Tarabasov, N. D. The determination of the stresses in a plate with several circular disks inserted in it.** Doklady Akad. Nauk SSSR (N.S.) 63, 15–18 (1948). (Russian)

A thin isotropic elastic plate occupies a finite multiply-connected region $S_0$ bounded by an exterior circular contour $\gamma_0$ of radius $r_0$ and $m$ symmetrically placed nonoverlapping interior circular holes $\gamma_i$ of radii $r_i$. Circular disks of the same material as the plate are forced into the holes $\gamma_i$, the initial radii $\rho_i$ of the disks being somewhat larger than $r_i$. What is the state of stress in a solid circular plate formed in this way, if the external forces are known along $\gamma_0$ and on the remaining contours $\gamma_i$ $(i=1, \cdots, m)$ the discontinuities of the displacement vector are given? The jump in the displacement vector on each contour $\gamma_i$ is equal to the difference $\rho_i-r_i$. This problem is solved by determining an appropriate set of analytic functions of a complex variable arising in the formulation of plane elastic problems by N. I. Mušhelišvili. *I. S. Sokolnikoff* (Los Angeles, Calif.).

**Budiansky, Bernard, and Hu, Pai C. The Lagrangian multiplier method of finding upper and lower limits to critical stresses of clamped plates.** Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **848**, 11 pp. (1946).

Reprint of Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1103** (1946); these Rev. **8**, 118.

identical with the paper on p. 653

**Silverj, Domenico Gentiloni. La integrazione della equazione $\Delta^4\chi=0$ in un problema elastico in tre dimensioni.** Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **206**, 44 pp. (1947).

The problem discussed is that of forcing a cap on to a rigid disc of diameter slightly greater than the internal diameter of the rim of the cap. Mathematically, this means solving a three-dimensional elastostatic problem, the displacement being given on the inside of the rim of the cap and the stress across the rest of the surface of the cap being zero. However, on account of the symmetry of the problem, the assignment of displacement is equivalent to assignment of a component of strain, and so (through the stress-strain relations) equivalent to a condition on the stress components. Thus the surface conditions are expressible entirely in terms of stress. The author uses the known fact that stresses expressed in terms of any axially symmetric three-dimensional biharmonic function $\chi$ satisfy the equations of equilibrium and compatibility. The purpose of the paper is to choose a biharmonic $\chi$ which makes the errors in the boundary conditions small. To this end there is taken as $n$th approximation $\chi=\sum_{i=1}^{n}C_i\chi_i$, where $\chi_i$ is a biharmonic polynomial of degree $i$ in cylindrical coordinates $\rho$, $z$, and $C_i$ are any constants. The error is computed by integrating over the surface of the cap the sum of the squares of the differences between the required stresses and those given by $\chi$, the constants $C_i$ being chosen to minimize the integral. At this stage arithmetical values are introduced and computations made up to $n=5$. [In the title of the paper, the equation should read $\Delta^2\chi=0$ rather than $\Delta^4\chi=0$.]
*J. L. Synge* (Dublin).

**Thomson, William T. The Laplace transform solution of beams.** J. Acoust. Soc. Amer. 21, 34–38 (1949).

The author employs the unilateral Laplace transform to solve some ordinary linear differential equations which arise in beam theory. A solution is given for the forced vibration of a hinged beam excited by a concentrated harmonic force. The study of beams with several masses and springs is considered. *A. E. Heins* (Pittsburgh, Pa.).

**Stone, W. M. Note on a paper by N. J. Durant.** Philos. Mag. (7) 39, 988–991 (1948).

The author gives an application of the generalized Laplace transformation to the problem of the continuous beam of $N$ equal spans. [Cf. Samuelson, Bull. Amer. Math. Soc. 52, 240 (1946).] *A. E. Heins* (Pittsburgh, Pa.).

**Biezeno, C. B., and Koch, J. J. Note on the buckling of a vertically submerged tube.** Appl. Sci. Research A. 1, 131–138 (1948).

The possibility is investigated that a long elastic tube vertically submerged in a fluid will buckle under its own weight. The cases treated are that for which (1) the tube is heavy and rests on the bottom, (2) the tube floats with a vertical orientation. It is concluded that in each case the buckling cannot occur except for extremely small cross-section. *G. F. Carrier* (Providence, R. I.).

**Federhofer, Karl. Berechnung der Drehschwingungen eines Kreiszylinders mit Berücksichtigung des Einflusses der Baustoffdämpfung und einer äusseren Flüssigkeitsreibung.** Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 156, 573–582 (1948).

The normal modes of rotary oscillation of a cylindrical rod of circular section are considered. Using cylindrical coordinates $r$, $\varphi$, $z$, the only nonzero displacement is $u_\varphi(r, z, t)$, and for this problem it takes the form $U(r)e^{i(\gamma z+pt)}$. Here $U(r)$ is determined from the equation of rotary motion, and the stress-strain relation $\tau_{ij}=\bar{G}\gamma_{ij}+\xi\dot{\gamma}_{ij}$ for the two nonzero stress components $\tau_{\varphi z}$ and $\tau_{r\varphi}$; $\bar{G}$ is complex and represents elasticity and hysteresis; $\xi$ represents internal friction. At the cylindrical surface fluid friction is considered by assuming a shear stress proportional to the velocity with a reversed sign. The ends are assumed unstressed. For this system the frequency equation becomes an intrinsic equation in Bessel functions with complex argument. An approximate solution is obtained for small hysteresis, internal and external friction, by expanding to the first order in these terms, for the mode of vibration in which the cross sections are not distorted but rotate relatively to one another. This solution agrees with that deduced from the wave equation in $z$ and $t$ for the transmission of torsional waves along the rod. *E. H. Lee* (Providence, R. I.).

**Prey, Adalbert. Über die Theorie der Landbrücken und die Viskosität der Erde.** Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 156, 593–624 (1 plate) (1948).

The author attempts to find a mechanism corresponding to the theory of land bridges across the Atlantic and Indian oceans. He considers the sinking of a round continent (radius 2000 km., thickness 30 or 50 km.) or of a circular

plate, which at the initial time lies on a viscous substratum. The solution of the problem is based on a solution of the simplified equations of motion given by H. Lamb [Hydrodynamics]. The continent is represented by an expression containing the first 16 spherical harmonics. A differential equation of motion is given and the depth and velocity of sinking are calculated for various coefficients of viscosity of the substratum. *W. Jardetzky* (Graz).

# MATHEMATICAL PHYSICS

## *Optics, Electromagnetic Theory*

**Hubert, Pierre. Lentille magnétique à axe curviligne.** C. R. Acad. Sci. Paris **228**, 302–304 (1949).

**Buchholz, Herbert. Die axialsymmetrische elektromagnetische Strahlung zwischen konfokalen Drehparabolen bei verschiedenen Anregungsarten.** Ann. Physik (6) **2**, 185–210 (1948).

The most general problem investigated in this paper is the electromagnetic field between two confocal paraboloids of revolution, the field being generated by a circular electric or magnetic current whose axis is the axis of revolution. A number of special problems, of more immediate interest in electrical theory, are also discussed: in these either the inner or the outer paraboloid is dispensed with, or else the circular current lies in the surface of one of them, or shrinks to a point of the axis of revolution. In every case, paraboloidal coordinates $(\xi, \eta, \varphi)$ are used and the solution is obtained by separating variables. The normal solutions with axial symmetry of the wave equation in paraboloidal coordinates are of the form $(\xi\eta)^{-\frac{1}{2}}W_{\pm s, \frac{1}{2}}(\pm 2ki\xi)W_{\pm s, \frac{1}{2}}(\mp 2ki\eta)$, where Whittaker's $W$-functions may be replaced by the $M$-functions. The general solution is an integral of normal solutions, multiplied by a suitable function of $s$ (but independent of $\xi$ and $\eta$) and integrated with respect to $s$ over a Mellin-Barnes type of path. The solutions of the various problems are represented by such integrals, and some of them are converted into infinite series by the calculus of residues. *A. Erdélyi* (Edinburgh).

**Davy, N. The electric field of a condenser of which one plate is an arc and the other a radius of a circular cylinder.** Philos. Mag. (7) **39**, 510–518 (1948).

The author investigates the electrical properties of a two-dimensional condenser of which one plate is a radius of a circular cylinder. The second plate is an arc of the same cylinder, on the side opposite the chosen radius and symmetric with respect to the end of the radius. If the complex potential is $W=U+iV$ the appropriate conformal transformation is the Weierstrassian elliptic function $z=\wp(AW)$, where $A$ is a constant. If we write the transformation in the alternative form

$$AW=\int_z^\infty [4(z-e_1)(z-e_2)(z-e_3)]^{-\frac{1}{2}}dz,$$

then one of the constants $e_2$ is real, corresponding to the center of the circle, while the other two are conjugate complex numbers, corresponding to each end of the circular arc. Equipotentials and lines of force are discussed, and derived formulas include the electric intensity, the surface charge density, the capacity and the mechanical force on a small dipole in the field. *M. C. Gray.*

**Cotte, Maurice. Potentiel et champ d'une électrode plane percée d'un trou elliptique.** C. R. Acad. Sci. Paris **228**, 377–378 (1949).

**Bertein, François. Théories non linéaires du champ électromagnétique.** Revue Sci. **86**, 349–356 (1948).
Survey article.

**Crout, Prescott D. An extension of Lagrange's equations to electromagnetic field problems.** J. Appl. Phys. **19**, 1007–1019 (1948).

The author's summary is, in part, as follows. This paper contains the basic theory of an extension of Lagrange's equations which renders these equations and the procedures associated with them in mechanics applicable to electromagnetic field problems such as are encountered in ultra-high-frequency work. In certain important cases these procedures give equivalent networks. Applications to several practical problems are indicated. *C. J. Bouwkamp.*

**Papas, C. H., and King, Ronold. Surface currents on a conducting sphere excited by a dipole.** J. Appl. Phys. **19**, 808–816 (1948).

For an array consisting of a linear antenna of height $h$ erected on the surface of a perfectly conducting sphere of radius $a$ the total current crossing a parallel of latitude $\theta$ on the sphere can be expressed as an infinite series of Legendre functions. It is assumed that the current distribution on the antenna is sinusoidal and that there is rotational symmetry about the antenna axis. Computation of the first twelve terms of the series for a quarter-wave-length antenna is described and curves are drawn of the current variation with $\theta$ for values of $a$ between $\lambda/8$ and $\lambda$ at intervals of $\lambda/8$. *M. C. Gray* (Murray Hill, N. J.).

**Romero Juárez, Antonio. Periods of motion in periodic orbits in the equatorial plane of a magnetic dipole.** Physical Rev. (2) **75**, 137–139 (1949).

The equation of the orbit of a charged particle moving in the equatorial plane of a dipole is expressed in terms of elliptic functions. In this presentation the periodic orbits can be selected and a table gives length of periods for orbits with different numbers of loops. *L. Jánossy* (Dublin).

## *Quantum Mechanics*

**Tzou, K. H. Relativistic Hamiltonian system of a particle and relativistic Heisenberg's equation.** Philos. Mag. (7) **39**, 790–799 (1948).

Heisenberg's commutation relations are written in relativistic form and are applied to a Dirac electron. Tensor operators for total angular momentum and spin are obtained. *A. Schild* (Toronto, Ont.).

**Gupta, S. A note on an analogy regarding operators in Dirac's wave equation for the electron.** Proc. Nat. Inst. Sci. India **14**, 311–314 (1948).

**Dyson, F. J. The radiation theories of Tomonaga, Schwinger, and Feynman.** Physical Rev. (2) **75**, 486–502 (1949).

The Tomonaga-Schwinger quantum electrodynamics is discussed with due emphasis on the physical ideas involved and the equivalence with a mainly unpublished theory by Feynman is established. The fundamental equation is $i\hbar c\partial\Omega/\partial\sigma(x_0)=H_T(x_0)\Omega$, where $H_T=(S(\sigma))^{-1}H^e(x_0)S(\sigma)$; here $H^e$ is the energy operator produced by external fields and $S$ satisfies a similar equation with $H_T$ replaced by $H^I=H^i+H^s$, the sum of the interaction energy and the electron self energy. Then a state with constant $\Omega$ is a system of electrons and photons moving under their mutual interactions with no external field present. In the Schwinger representation the matrix element between two states is $\Omega_2^*H_T\Omega_1$; in Feynman's theory symmetry between past and future is provided by reversing the direction of time in $\Omega$ to define a state vector $\Omega'=S(\infty)\Omega(\sigma)$ and taking the matrix element of $H_T$ between two states as $\Omega_2'^*H_F\Omega_1$, where $H_F(x_0)=S(\infty)H_T(x_0)$. It is established that $H_F$ is a function of $x_0$ alone and not of the surface $\sigma$. A set of rules is given for calculating the matrix elements of $H_F$ between two given states and the results simplified by using a graphical representation. This graphical method is illustrated by applying it to the calculation of the second-order radiative corrections to the motion of an electron in an external field.

*H. C. Corben* (Pittsburgh, Pa.).

**Koba, Zirô, Ôisi, Yasuharu, and Sasaki, Muneo. Auxiliary condition and gauge transformation in the "super-many-time theory." II.** Progress Theoret. Physics **3**, 229–243 (1948).

This paper is a continuation of an earlier discussion of the auxiliary condition [same vol., 141–151 (1948); these Rev. **10**, 226]. Here a generating functional for a gauge transformation is obtained for the theory of the interaction of photons with other fields, using the Heisenberg picture as before, and the invariance of the formulations of meson photon-fields under the transformations derived from this functional investigated. A transformation is given under which both meson and electromagnetic quantities are changed, but the Tomonaga-Schroedinger equation is not invariant; however, it is possible to define a transformation in which the meson quantities and this equation are invariant while the electromagnetic potentials undergo a gauge transformation. *H. C. Corben* (Pittsburgh, Pa.).

**de Wet, J. S. On the relativistic invariance of quantized field theories.** Proc. Roy. Soc. London. Ser. A. **195**, 365–376 (1948).

The purpose of this paper is to show that the quantization of linear field equations derived from higher order Lagrangians [Proc. Cambridge Philos. Soc. **44**, 546–559 (1948); these Rev. **10**, 91] is relativistically invariant under the full group of coordinate transformations. The method is based on Weiss's use of generalized Poisson brackets [Proc. Roy. Soc. London. Ser. A. **169**, 102–119 (1938)] the invariance of which under canonical transformations leads to the relativistic invariance of the quantized Fermi-Dirac and Bose-Einstein theories. However, in the case of Lagrangians involving higher order derivatives the order of covariant differentiation is important, so that in treating this case it is assumed that the space-time is flat. The general invariance of the classical and quantized field equations subject to this restriction is then established, by first showing that the Weiss brackets are invariant even for Lagrangians involving higher derivatives. *H. C. Corben.*

**Schönberg, Mario. Elimination of divergences in the meson theory. I.** Physical Rev. (2) **74**, 748–760 (1948).

This paper applies to meson theory, in particular the charged pseudoscalar version, methods developed by the author for the classical and quantum theory of the point electron [same Rev. (2) **69**, 211–224 (1946); **74**, 738–747 (1948); Anais Acad. Brasil. Ci. **18**, 297–339 (1946); these Rev. **8**, 428, 615; and a forthcoming paper]. From the author's summary: ". . . The formalism, associated to special rules of interpretation, leads to finite transition probabilities; it can also be applied to self-energy problems and leads to finite self-energies and magnetic moments. In the computation of the self-energies and magnetic moments two constants appear, which may be chosen in such a way as to give different masses to the proton and neutron."

*L. Hulthén* (Stockholm).

**Balseiro, José A. Concerning a canonical transformation of the radiation field.** Revista Unión Mat. Argentina **13**, 106–119 (1948). (Spanish. English summary)

It is shown that the canonical transformation, which leads from the variables $p, q$ of a linear oscillator to the new variables $N$, $\vartheta$ ($N$, number of excitations, $\vartheta$ phase) does not fit with the ordinary scheme of transformation theory. A generalization of this scheme, suitable for the present case, is given. *Author's summary.*

**Rădulet, Remus. Sur les tensions fictives équivalentes aux actions pondéromotrices des champs physiques.** Bull. Sci. Tech. Polytech. Timişoara **13**, 54–68 (1948).

**Fischer, Otto F. Why not discard the spinor calculus?** Philos. Mag. (7) **39**, 878–884 (1948).

The argument the author gives for discarding the spinor calculus may be summarized as follows. A second order differential equation similar to that derived from the Dirac theory may be written in terms of quaternions. He therefore claims that the quaternion calculus is as well (if not better) suited for the discussion of quantum mechanics as is the spinor calculus. He does not seem to be aware of the need for discussing the transformation properties of the wave functions of quantum theory under Lorentz transformations, nor does he remark that these do not always behave as vectors and tensors, but they do transform in accordance with representations of the Lorentz group which may be double-valued. The spinor calculus is formalism designed to handle these representations and the manner in which it includes the quaternion calculus is known.

*A. H. Taub* (Urbana, Ill.).

**Murard, Robert. Sur une théorie générale des corpuscules et des systèmes de corpuscules et ses applications.** Ann. Physique (12) **3**, 550–619 (1948).

In the first part of this paper, a set of postulates for free particles is given. From these it is shown that the wave equation describing elementary particles is the Dirac equation or a similar one in which the mass term is represented by an operator having proper values not all equal. In the second part of the paper some of these postulates are modified so that a system of particles can be treated. A discussion of the interaction of particles of various kinds is given. *A. H. Taub* (Urbana, Ill.).

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